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Simultaneous Approximation of Sobolev Classes by Piecewise Cubic Hermite Interpolation

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Abstract. For the approximation in L_p-norm, we determine the weakly asymptotic orders for the simultaneous approximation errors of Sobolev classes by piecewise cubic Hermite interpolation with equidistant knots. For $p = 1, \infty$, we obtain its values. By these results we know that for the Sobolev classes, the approximation errors by piecewise cubic Hermite interpolation are weakly equivalent to the corresponding infinite-dimensional Kolmogorov widths. At the same time, the approximation errors of derivatives are weakly equivalent to the corresponding infinite-dimensional Kolmogorov widths.

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Key words: Piecewise cubic Hermite interpolation, L_p-norm, simultaneous approximation, equidistant knot, infinite-dimensional Kolmogorov width.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ be the set of all positive integers, all integers and all real numbers, respectively. For $1 \leq p \leq \infty$, let L_p be the spaces of functions $f : \mathbb{R} \to \mathbb{R}$ with the corresponding norms $\|\cdot\|_p$. Denote by $W_p^r(\mathbb{R})$, $r \in \mathbb{N}$, the class of functions f such that $f^{(r-1)}(f^{(0)} := f)$ is locally absolutely continuous and $\|f^{(r)}\|_p \leq 1$.

The approximation of periodic Sobolev classes by periodic polynomial splines with restrictions on its derivatives has been studied for a long time (see [1–4, 7–9]). In these researches, the approximation polynomial splines are assumed with equidistant knots and with defect 1. Recently, [14] and [15] consider the approximation of non-periodic Sobolev classes by polynomial splines with restrictions, where the approximation polynomial splines are also assumed with equidistant knots and with defect 1.

The simultaneous approximation problems for smooth functions are an important research topic in approximation theory and application. For polynomial simultaneous

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approximation problem, the main results can be found in [6] and [12]. As to the concrete interpolation polynomial operators, the main results can be looked up in [11] and [13]. As far as we know, all relevant results are only connected to a single function approximation and not for function classes approximation. In [1–4, 7–9, 14, 15], all results are connected to function approximation only, however the simultaneous approximation problems can also be discussed since both the functions and approximation splines have derivatives up to r-order. Hence, we want to consider the simultaneous approximation of Sobolev classes by piecewise cubic Hermite interpolation with equidistant knots. It is well known that the defect of piecewise cubic Hermite interpolation is 2.

Now we give the definition of piecewise cubic Hermite interpolation on knots $x_k = k/n$, $k \in \mathbb{Z}$. For $f \in C^{(1)}(\mathbb{R})$, there is an unique piecewise cubic polynomial $H_n(f, x)$ with knots $x_k = k/n$, $k \in \mathbb{Z}$ and satisfies the following conditions:

(1). $H_n(f, x) \in C^{(1)}(\mathbb{R})$;

(2). $H_n(f, x_k) = f(x_k)$, $H_n'(f, x_k) = f'(x_k)$, $k \in \mathbb{Z}$;

(3). $H_n(f, x)$ is a cubic polynomial about x on each subinterval $[x_{k-1}, x_k]$, $k \in \mathbb{Z}$.

It is well known that for $x \in [x_k, x_{k+1}]$,

$$\begin{aligned} H_n(f,x) =& f(x_k)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2\left(1+\frac{2(x-x_k)}{x_{k+1}-x_k}\right) \\ &+ f'(x_k)(x-x_k)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2 \\ &+ f(x_{k+1})\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2\left(1+\frac{2(x-x_{k+1})}{x_k-x_{k+1}}\right) \\ &+ f'(x_{k+1})(x-x_{k+1})\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2. \end{aligned} \tag{1.1}$$

On the one hand, we will consider the second derivative approximation of piecewise cubic Hermite interpolation on Sobolev classes $W_p^3(\mathbb{R})$. We obtain the following results.

Theorem 1.1. *Let $H_n(f, x)$ be defined as (1.1). Then we have*

$$\sup_{f\in W_\infty^3(\mathbb{R})} \|H_n''f - f''\|_\infty = \frac{8}{27n}, \tag{1.2}$$

$$\sup_{f\in W_1^3(\mathbb{R})} \|H_n''f - f''\|_1 = \frac{C_1}{n}, \tag{1.3}$$

and for $1 < p < \infty$,

$$\sup_{f\in W_p^3(\mathbb{R})} \|H_n''f - f''\|_p \le \left(\frac{8}{27}\right)^{1-\frac{1}{p}} C_1^{\frac{1}{p}} \frac{1}{n}. \tag{1.4}$$

Where C_1 is defined in (3.30).

On the other hand, we will consider the first derivative approximation of piecewise cubic Hermite interpolation on Sobolev classes $W_p^3(\mathbb{R})$. Our results are as follows.

Theorem 1.2. *Let* $H_n(f,x)$ *be defined as* (1.1). *Then we have*

$$\sup_{f\in W_\infty^3(\mathbb{R})} \|H_n'f - f'\|_\infty = \frac{13\sqrt{13}-46}{27n^2}, \tag{1.5}$$

$$\sup_{f\in W_1^3(\mathbb{R})} \|H_n'f - f'\|_1 = \frac{13\sqrt{13}-46}{27n^2}, \tag{1.6}$$

and for $1 < p < \infty$*,*

$$\sup_{f\in W_p^3(\mathbb{R})} \|H_n'f - f'\|_p \le \frac{13\sqrt{13}-46}{27n^2}. \tag{1.7}$$

At last, we will consider the function approximation of piecewise cubic Hermite interpolation on Sobolev classes $W_p^3(\mathbb{R})$. We get the following results.

Theorem 1.3. *Let* $H_n(f,x)$ *be defined as* (1.1). *Then we have*

$$\sup_{f\in W_\infty^3(\mathbb{R})} \|H_nf - f\|_\infty = \frac{1}{96n^3}, \tag{1.8}$$

$$\sup_{f\in W_1^3(\mathbb{R})} \|H_nf - f\|_1 = \frac{\sqrt{3}}{216n^3}, \tag{1.9}$$

and for $1 < p < \infty$*,*

$$\sup_{f\in W_p^3(\mathbb{R})} \|H_nf - f\|_p \le \left(\frac{1}{96}\right)^{1-\frac{1}{p}} \left(\frac{\sqrt{3}}{216}\right)^{\frac{1}{p}} \frac{1}{n^3}. \tag{1.10}$$

Remark 1.1. Suppose S is a space of functions defined on $\mathbb{R}$. The space S is called an infinite-σ-dimensional space if there is a real number $\sigma \in \mathbb{R}_+$ such that

$$\widetilde{\dim}(S) := \lim_{\alpha\to\infty} \inf \frac{\dim S|_{[-a,a]}}{2\alpha} = \sigma,$$

where $S|_{[-a,a]}$ is the space consisting of functions in S restricted on $[-a,a]$ and $\dim S|_{[-a,a]}$ is the dimension of $S|_{[-a,a]}$. The quality

$$\widetilde{d}_\sigma(W_p^r(\mathbb{R}))_p := \inf_{\widetilde{\dim}(S)\le\sigma} \sup_{f\in W_p^r(\mathbb{R})} \inf_{g\in S} \|f-g\|_p$$

is called the infinite-σ-dimensional Kolmogorov width of $W_p^r(\mathbb{R})$ in $L_p(\mathbb{R})$.

It is easy to know that the space of spline functions $S = \{H_n(f,x)|f \in C^{(1)}(\mathbb{R})\}$ satisfies

$$\widetilde{\dim}(S) = 2n.$$

At the same time, from Theorem 1.3 and [10], we know that for $1 \le p \le \infty$,

$$\sup_{f\in W_p^3(\mathbb{R})} \|H_n f - f\|_p \asymp \widetilde{d}_{2n}(W_p^3(\mathbb{R}))_p \asymp \frac{1}{n^3}.$$

Here and in the following the notation $a_n \asymp b_n$ for sequences $\{a_n\}$ and $\{b_n\}$ of positive numbers means the existence of a positive constant C independent of n such that $a_n/C \le b_n \le Ca_n$, and constants C may be different in the different expressions.

Besides, from the definition of Sobolev classes, we know that for $1 \le p \le \infty$,

$$\big\{f'|f \in W_p^3(\mathbb{R})\big\} = W_p^2(\mathbb{R}), \quad \big\{f''|f \in W_p^3(\mathbb{R})\big\} = W_p^1(\mathbb{R}).$$

From Theorem 1.2, Theorem 1.1 and [10], we obtain that for $1 \le p \le \infty$,

$$\sup_{f\in W_p^3(\mathbb{R})} \|H_n' f - f'\|_p \asymp \widetilde{d}_{2n}(W_p^2(\mathbb{R}))_p \asymp \frac{1}{n^2},$$

$$\sup_{f\in W_p^3(\mathbb{R})} \|H_n'' f - f''\|_p \asymp \widetilde{d}_{2n}(W_p^1(\mathbb{R}))_p \asymp \frac{1}{n}.$$

Remark 1.2. Using the same method, we can obtain the corresponding simultaneous approximation results of periodic Sobolev classes (or Sobolev classes on a closed interval) by piecewise cubic Hermite interpolation with equidistant knots.

2. Some lemmas

Let T be a linear bounded mapping from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$. The norm of the mapping T is defined as

$$\|T\|_p = \sup_{f\neq 0} \frac{\|Tf\|_p}{\|f\|_p}. \tag{2.1}$$

From the well-known Riesz-Thorin interpolation theorem (see [5]), we obtain the following lemma.

Lemma 2.1. *Let T be a linear bounded mapping from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$ for $1 \le p \le +\infty$. Then for $1 < p < \infty$,*

$$\|T\|_p \le \|T\|_1^{\frac{1}{p}} \|T\|_\infty^{1-\frac{1}{p}}. \tag{2.2}$$

Let $K(x,t)$ be a bounded measurable function on $[0,h] \times [0,h]$ and let

$$T(f,x) = \int_0^h f(t)K(x,t)dt. \tag{2.3}$$

The following lemma is well known.

Lemma 2.2. *Let $K(x,t)$ be a piecewise continuous function on $[0,h]\times[0,h]$ and let T be defined by* (2.3). *Then T is a linear bounded mapping from $L_p[0,h]$ to $L_p[0,h]$ for $1\le p\le\infty$, and*

$$\|T\|_\infty = \max_{0\le x\le h}\int_0^h |K(x,t)|dt, \tag{2.4}$$

$$\|T\|_1 = \max_{0\le t\le h}\int_0^h |K(x,t)|dx. \tag{2.5}$$

3. Proof of Theorem 1.1

Proof. Denote $h=1/n$. From (1.1) we know: if $x\in[x_{k-1},x_k]$, $k\in\mathbb{Z}$, then

$$\begin{aligned} H_n(f,x) =& \frac{f(x_{k-1})(x-x_k)^2}{h^2}\left(1+\frac{2(x-x_{k-1})}{h}\right)+\frac{f'(x_{k-1})(x-x_{k-1})(x-x_k)^2}{h^2} \\ &+\frac{f(x_k)(x-x_{k-1})^2}{h^2}\left(1-\frac{2(x-x_k)}{h}\right)+\frac{f'(x_k)(x-x_{k-1})^2(x-x_k)}{h^2}. \end{aligned} \tag{3.1}$$

For $k=1$ and $x\in[0,h]$, by (3.1), it follows that

$$\begin{aligned} H_n(f,x) =& \frac{f(0)(x-h)^2}{h^2}\left(1+\frac{2x}{h}\right)+\frac{f'(0)x(x-h)^2}{h^2} \\ &+\frac{f(h)x^2}{h^2}\left(3-\frac{2x}{h}\right)+\frac{f'(h)x^2(x-h)}{h^2}. \end{aligned} \tag{3.2}$$

Differentiating two times on both hands of (3.2) and applying the Newton-Leibniz formula, we obtain

$$\begin{aligned} H_n''(f,x) &= \frac{f(0)\,(12x-6h)}{h^3}+\frac{f'(0)(6x-4h)}{h^2}+\frac{f(h)(6h-12x)}{h^3}+\frac{f'(h)(6x-2h)}{h^2} \\ &= \frac{6h-12x}{h^3}\int_0^h f'(t)dt+\frac{f'(0)(6x-4h)}{h^2}+\frac{f'(h)(6x-2h)}{h^2} \\ &= \frac{1}{h^3}\int_0^h \big[(6h-12x)f'(t)+f'(0)(6x-4h)+f'(h)(6x-2h)\big]\,dt \\ &= \frac{4h-6x}{h^3}\int_0^h \big[f'(t)-f'(0)\big]\,dt+\frac{6x-2h}{h^3}\int_0^h \big[f'(h)-f'(t)\big]\,dt \\ &= \frac{4h-6x}{h^3}\int_0^h dt\int_0^t f''(s)ds+\frac{6x-2h}{h^3}\int_0^h dt\int_t^h f''(s)ds. \end{aligned} \tag{3.3}$$

Exchanging the integral order, we obtain

$$\int_0^h dt\int_0^t f''(s)ds=\int_0^h ds\int_s^h f''(s)dt=\int_0^h (h-s)f''(s)ds. \tag{3.4}$$

Similarly, one has

$$\int_0^h dt \int_t^h f''(s)ds = \int_0^h sf''(s)ds. \tag{3.5}$$

From (3.3), (3.4) and (3.5), it follows that

$$H_n''(f,x) = \frac{1}{h^3}\int_0^h [4h^2 - 6hs - 6xh + 12xs]f''(s)ds. \tag{3.6}$$

It is easy to verify that

$$\frac{1}{h^3}\int_0^h [4h^2 - 6hs - 6xh + 12xs]ds = 1. \tag{3.7}$$

From (3.6), (3.7) and the Newton-Leibniz formula, we obtain

$$\begin{aligned} H_n''(f,x) - f''(x) =& \frac{1}{h^3}\int_0^h [4h^2 - 6hs - 6xh + 12xs]\left[f''(s) - f''(x)\right] ds \\ =& \frac{1}{h^3}\int_0^h [4h^2 - 6hs - 6xh + 12xs]ds \int_x^s f^{(3)}(t)dt \\ =& -\frac{1}{h^3}\int_0^x [4h^2 - 6hs - 6xh + 12xs]ds \int_s^x f^{(3)}(t)dt \\ &+ \frac{1}{h^3}\int_x^h [4h^2 - 6hs - 6xh + 12xs]ds \int_x^s f^{(3)}(t)dt. \end{aligned} \tag{3.8}$$

Exchanging the integral order, we obtain

$$\begin{aligned} &\int_0^x [4h^2 - 6hs - 6xh + 12xs]ds \int_s^x f^{(3)}(t)dt \\ =& \int_0^x f^{(3)}(t)dt \int_0^t [4h^2 - 6hs - 6xh + 12xs]ds \\ =& \int_0^x f^{(3)}(t)\left[4h^2t - 3ht^2 - 6xht + 6xt^2\right] dt. \end{aligned} \tag{3.9}$$

Similarly, one has

$$\begin{aligned} &\int_x^h [4h^2 - 6hs - 6xh + 12xs]ds \int_x^s f^{(3)}(t)dt \\ =& \int_x^h f^{(3)}(t)dt \int_t^h [4h^2 - 6hs - 6xh + 12xs]ds \\ =& \int_x^h f^{(3)}(t)\left[h^3 - 4h^2t + 3ht^2 + 6xht - 6xt^2\right] dt. \end{aligned} \tag{3.10}$$

Denote

$$K_1(x,t) = \begin{cases} \frac{-4h^2t+3ht^2+6xht-6xt^2}{h^3}, & 0 \le t \le x \le h; \\ \frac{h^3-4h^2t+3ht^2+6xht-6xt^2}{h^3}, & 0 \le x \le t \le h. \end{cases} \tag{3.11}$$

Then by (3.8)-(3.11), we know

$$H_n''(f,x)-f''(x)=\int_0^h f^{(3)}(t)K_1(x,t)dt. \tag{3.12}$$

For $p=\infty$, from (2.4) and (3.12), it follows that

$$\sup_{f\in W^3_\infty(\mathbb{R})}\max_{0\le x\le h}|H_n''(f,x)-f''(x)|=\max_{0\le x\le h}\int_0^h|K_1(x,t)|dt. \tag{3.13}$$

From (3.11), it follows that for an arbitrary $0\le x\le h$,

$$\begin{aligned}\int_0^h|K_1(x,t)|dt=&\frac{1}{h^3}\int_0^x\left|4h^2t-3ht^2-6xht+6xt^2\right|dt\\ &+\frac{1}{h^3}\int_x^h\left|h^3-4h^2t+3ht^2+6xht-6xt^2\right|dt.\end{aligned} \tag{3.14}$$

We consider the first integral in (3.14) now. For $0\le x\le \frac{2h}{3}$, it is easy to verify that

$$\begin{aligned}&\int_0^x\left|4h^2t-3ht^2-6xht+6xt^2\right|dt\\ =&\int_0^x\left[4h^2t-3ht^2-6xht+6xt^2\right]dt=2x^2(h-x)^2.\end{aligned} \tag{3.15}$$

For $\frac{2h}{3}\le x\le h$, by a direct computation, we obtain

$$\begin{aligned}&\int_0^x\left|4h^2t-3ht^2-6xht+6xt^2\right|dt\\ =&-\int_0^{\frac{2h(2h-3x)}{3h-6x}}\left[4h^2t-3ht^2-6xht+6xt^2\right]dt\\ &+\int_{\frac{2h(2h-3x)}{3h-6x}}^x\left[4h^2t-3ht^2-6xht+6xt^2\right]dt\\ =&2x^2(h-x)^2+\frac{8h^3(3x-2h)^3}{27(h-2x)^2}.\end{aligned} \tag{3.16}$$

We will consider the second integral in (3.14). For $0\le x\le\frac{h}{3}$, it is easy to verify that

$$\begin{aligned}&\int_x^h\left|h^3-4h^2t+3ht^2+6xht-6xt^2\right|dt\\ =&\int_x^{\frac{h^2}{3h-6x}}\left[h^3-4h^2t+3ht^2+6xht-6xt^2\right]dt\\ &-\int_{\frac{h^2}{3h-6x}}^h\left[h^3-4h^2t+3ht^2+6xht-6xt^2\right]dt\\ =&2x^2(h-x)^2+\frac{8h^3(h-3x)^3}{27(h-2x)^2}.\end{aligned} \tag{3.17}$$

For $\frac{h}{3} \le x \le h$, by a simple computation, it follows that

$$\int_x^h \left|h^3 - 4h^2t + 3ht^2 + 6xht - 6xt^2\right| dt = 2x^2(h-x)^2. \tag{3.18}$$

Combining (3.14)-(3.18), we obtain

$$\int_0^h |K_1(x,t)|dt = \begin{cases} \frac{4}{h^3}\left[x^2(h-x)^2 + \frac{2h^3(h-3x)^3}{27(h-2x)^2}\right], & 0 \le x \le \frac{h}{3}; \\ \frac{4x^2(h-x)^2}{h^3}, & \frac{h}{3} \le x \le \frac{2h}{3}; \\ \frac{4}{h^3}\left[x^2(h-x)^2 + \frac{2h^3(3x-2h)^3}{27(h-2x)^2}\right], & \frac{2h}{3} \le x \le h. \end{cases} \tag{3.19}$$

By (3.19) and the computation of maximal value of differentiable functions on a closed interval, we get

$$\max_{0\le x\le h} \int_0^h |K_1(x,t)|dt = \int_0^h |K_1(h,t)|dt = \frac{8h}{27}. \tag{3.20}$$

From (3.13) and (3.20), it follows that

$$\sup_{f\in W_\infty^3(\mathbb{R})} \max_{0\le x\le h} |H_n''(f,x) - f''(x)| = \frac{8h}{27}. \tag{3.21}$$

Similarly, for an arbitrary $k \in \mathbb{Z}$, we have

$$\sup_{f\in W_\infty^3(\mathbb{R})} \max_{(k-1)h\le x\le kh} |H_n''(f,x) - f''(x)| = \frac{8h}{27}. \tag{3.22}$$

From (3.22) we yield (1.2).

For $p = 1$, from (2.5) and (3.12), it follows that for $f \in W_1^3(\mathbb{R})$,

$$\int_0^h \left|H_n''(f,x) - f''(x)\right| dx \le \max_{0\le t\le h} \int_0^h |K_1(x,t)|dx \cdot \int_0^h \left|f^{(3)}(t)\right| dt. \tag{3.23}$$

From (3.11) we know that for $0 \le t \le h$, we have

$$\begin{aligned}\int_0^h |K_1(x,t)|dx =& \frac{1}{h^3}\int_t^h \left|4h^2t - 3ht^2 - 6xht + 6xt^2\right| dx \\ &+ \frac{1}{h^3}\int_0^t \left|h^3 - 4h^2t + 3ht^2 + 6xht - 6xt^2\right| dx. \end{aligned}\tag{3.24}$$

It is easy to verify that for $0 \le t \le \frac{2h}{3}$,

$$\begin{aligned}
&\int_t^h \left|4h^2t - 3ht^2 - 6xht + 6xt^2\right| dx \\
=&\int_t^{\frac{4h^2-3ht}{6(h-t)}} \left(4h^2t - 3ht^2 - 6xht + 6xt^2\right) dx \\
&- \int_{\frac{4h^2-3ht}{6(h-t)}}^h \left(4h^2t - 3ht^2 - 6xht + 6xt^2\right) dx \\
=&\frac{t(4h^2-3ht)^2}{6(h-t)} - 4h^2t^2 + 6ht^3 - 3t^4 - h^3t,
\end{aligned} \tag{3.25}$$

and for $\frac{2h}{3} \le t \le h$,

$$\begin{aligned}
\int_t^h \left|4h^2t - 3ht^2 - 6xht + 6xt^2\right| dx &= \int_t^h \left(4h^2t - 3ht^2 - 6xht + 6xt^2\right) dx \\
&= h^3t - 4h^2t^2 + 6ht^3 - 3t^4.
\end{aligned} \tag{3.26}$$

Similarly, we have that for $0 \le t \le \frac{h}{3}$,

$$\begin{aligned}
\int_0^t \left|h^3 - 4h^2t + 3ht^2 + 6xht - 6xt^2\right| dx &= \int_0^t \left(h^3 - 4h^2t + 3ht^2 + 6xht - 6xt^2\right) dx \\
&= h^3t - 4h^2t^2 + 6ht^3 - 3t^4,
\end{aligned} \tag{3.27}$$

and for $\frac{h}{3} \le t \le h$,

$$\begin{aligned}
&\int_0^t \left|h^3 - 4h^2t + 3ht^2 + 6xht - 6xt^2\right| dx \\
=&-\int_0^{\frac{h(3t-h)}{6t}} \left(h^3 - 4h^2t + 3ht^2 + 6xht - 6xt^2\right) dx \\
&+ \int_{\frac{h(3t-h)}{6t}}^t \left(h^3 - 4h^2t + 3ht^2 + 6xht - 6xt^2\right) dx \\
=&h^3t - 4h^2t^2 + 6ht^3 - 3t^4 + \frac{h^2(h-t)(3t-h)^2}{6t}.
\end{aligned} \tag{3.28}$$

From (3.24)-(3.28), it follows that

$$\begin{aligned}
&\int_0^h |K_1(x,t)| dx \\
=&\begin{cases}
\frac{1}{h^3}\left[\frac{t(4h^2-3ht)^2}{6(h-t)} - 8h^2t^2 + 12ht^3 - 6t^4\right], & 0 \le t \le \frac{h}{3}; \\
\frac{1}{h^3}\left[\frac{t(4h^2-3ht)^2}{6(h-t)} - 8h^2t^2 + 12ht^3 - 6t^4 + \frac{h^2(h-t)(3t-h)^2}{6t}\right], & \frac{h}{3} \le t \le \frac{2h}{3}; \\
\frac{1}{h^3}\left[2h^3t - 8h^2t^2 + 12ht^3 - 6t^4 + \frac{h^2(h-t)(3t-h)^2}{6t}\right], & \frac{2h}{3} \le t \le h.
\end{cases}
\end{aligned} \tag{3.29}$$

By (3.29) and a numerical solution we obtain that

$$C_1 = \frac{\max_{0\le t\le h}\int_0^h |K_1(x,t)|dx}{h} = 0.251498. \tag{3.30}$$

By (3.23) and (3.30), we know

$$\int_0^h \left|H_n''(f,x) - f''(x)\right| dx \le C_1 h \int_0^h \left|f^{(3)}(t)\right| dt. \tag{3.31}$$

Similar to the proof of (3.31), for an arbitrary $k \in \mathbb{Z}$, we have

$$\int_{(k-1)h}^{kh} \left|H_n''(f,x) - f''(x)\right| dx \le C_1 h \int_{(k-1)h}^{kh} \left|f^{(3)}(t)\right| dt. \tag{3.32}$$

Hence, (3.32) gives that for $f \in W_1^3(\mathbb{R})$,

$$\begin{aligned} \|H_n''f - f''\|_1 &= \sum_{k\in\mathbb{Z}} \int_{(k-1)h}^{kh} \left|H_n''(f,x) - f''(x)\right| dx \\ &\le C_1 h \sum_{k\in\mathbb{Z}} \int_{(k-1)h}^{kh} \left|f^{(3)}(t)\right| dt = C_1 h \left\|f^{(3)}\right\|_1 \le C_1 h. \end{aligned} \tag{3.33}$$

On the other hand, denote $\overline{W}_1^3(\mathbb{R}) = \left\{f \in W_1^3(\mathbb{R}) \middle| \operatorname{supp} f^{(3)} \subset [0,h]\right\}$. Then by (3.12), (2.5) and (3.30), we obtain

$$\sup_{f\in\overline{W}_1^3(\mathbb{R})} \|H_n''f - f''\|_1 = C_1 h. \tag{3.34}$$

From $\overline{W}_1^3(\mathbb{R}) \subset W_1^3(\mathbb{R})$, (3.33) and (3.34) we get (1.3). By (2.2), (1.2) and (1.3) we obtain (1.4). ☐

4. Proof of Theorem 1.2

Proof. For $x \in [0,h]$, from $H_n'(f,0) = f'(0)$, Newton-Leibniz formula and (3.8)-(3.10), it follows that

$$\begin{aligned} H_n'(f,x) - f'(x) &= \int_0^x \left[H_n''(f,t) - f''(t)\right] dt \\ &= -\frac{1}{h^3}\int_0^x dt \int_0^t f^{(3)}(s)\left[4h^2 s - 3hs^2 - 6ths + 6ts^2\right] ds \\ &\quad + \frac{1}{h^3}\int_0^x dt \int_t^h f^{(3)}(s)\left[h^3 - 4h^2 s + 3hs^2 + 6ths - 6ts^2\right] ds. \end{aligned} \tag{4.1}$$

Exchanging the integral order, we obtain

$$\begin{aligned}
&\int_0^x dt \int_0^t f^{(3)}(s)\left[4h^2s - 3hs^2 - 6ths + 6ts^2\right] ds \\
=&\int_0^x f^{(3)}(s)ds \int_s^x \left[4h^2s - 3hs^2 - 6ths + 6ts^2\right] dt \\
=&\int_0^x f^{(3)}(s)\left[4h^2xs - 3hxs^2 - 3hx^2s + 3x^2s^2 - 4h^2s^2 + 6hs^3 - 3s^4\right] ds. \qquad (4.2)
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
&\int_0^x dt \int_t^h f^{(3)}(s)\left[h^3 - 4h^2s + 3hs^2 + 6ths - 6ts^2\right] ds \\
=&\int_0^x dt \int_t^x f^{(3)}(s)\left[h^3 - 4h^2s + 3hs^2 + 6ths - 6ts^2\right] ds \\
&+\int_0^x dt \int_x^h f^{(3)}(s)\left[h^3 - 4h^2s + 3hs^2 + 6ths - 6ts^2\right] ds \\
=&\int_0^x f^{(3)}(s)ds \int_0^s \left[h^3 - 4h^2s + 3hs^2 + 6ths - 6ts^2\right] dt \\
&+\int_x^h f^{(3)}(s)ds \int_0^x \left[h^3 - 4h^2s + 3hs^2 + 6ths - 6ts^2\right] dt \\
=&\int_0^x f^{(3)}(s)\left[h^3s - 4h^2s^2 + 6hs^3 - 3s^4\right] ds \\
&+\int_x^h f^{(3)}(s)\left[h^3x - 4h^2xs + 3hxs^2 + 3hx^2s - 3x^2s^2\right] ds. \qquad (4.3)
\end{aligned}$$

Denote

$$K_2(x,s) = \begin{cases} \frac{h^3s-4h^2xs+3hxs^2+3hx^2s-3x^2s^2}{h^3}, & 0 \le s \le x \le h; \\ \frac{h^3x-4h^2xs+3hxs^2+3hx^2s-3x^2s^2}{h^3}, & 0 \le x \le s \le h. \end{cases} \qquad (4.4)$$

By (4.1)-(4.4), we know

$$H_n'(f,x) - f'(x) = \int_0^h f^{(3)}(s)K_2(x,s)ds. \qquad (4.5)$$

For $p=\infty$, from (2.4) and (4.5), it follows that

$$\sup_{f\in W_\infty^3(\mathbb{R})} \max_{0\le x\le h} |H_n'(f,x) - f'(x)| = \max_{0\le x\le h} \int_0^h |K_2(x,s)|ds. \qquad (4.6)$$

From (4.4), it follows that for an arbitrary $0 \le x \le h$,

$$\begin{aligned}
\int_0^h |K_2(x,s)|ds =& \frac{1}{h^3}\int_0^x \left|h^3s - 4h^2xs + 3hxs^2 + 3hx^2s - 3x^2s^2\right| ds \\
&+ \frac{1}{h^3}\int_x^h \left|h^3x - 4h^2xs + 3hxs^2 + 3hx^2s - 3x^2s^2\right| ds. \qquad (4.7)
\end{aligned}$$

We consider the first integral in (4.7) now. For $0\leq x\leq \frac{h}{3}$, it is easy to verify that

$$\int_0^x \left|h^3s-4h^2xs+3hxs^2+3hx^2s-3x^2s^2\right|ds=\frac{x^2(h-x)^2(h-2x)}{2}. \tag{4.8}$$

For $\frac{h}{3}\leq x\leq h$, by a direct computation, we obtain

$$\begin{aligned}&\int_0^x \left|h^3s-4h^2xs+3hxs^2+3hx^2s-3x^2s^2\right|ds\\ =&-\int_0^{\frac{h(3x-h)}{3x}}\left[h^3s-4h^2xs+3hxs^2+3hx^2s-3x^2s^2\right]ds\\ &+\int_{\frac{h(3x-h)}{3x}}^x\left[h^3s-4h^2xs+3hxs^2+3hx^2s-3x^2s^2\right]ds\\ =&\frac{x^2(h-x)^2(h-2x)}{2}+\frac{h^3(h-x)(3x-h)^3}{27x^2}.\end{aligned} \tag{4.9}$$

We will consider the second integral in (4.7). For $0\leq x\leq \frac{2h}{3}$, it is easy to verify that

$$\begin{aligned}&\int_x^h \left|h^3x-4h^2xs+3hxs^2+3hx^2s-3x^2s^2\right|ds\\ =&\int_x^{\frac{h^2}{3(h-x)}}\left[h^3x-4h^2xs+3hxs^2+3hx^2s-3x^2s^2\right]ds\\ &-\int_{\frac{h^2}{3(h-x)}}^h\left[h^3x-4h^2xs+3hxs^2+3hx^2s-3x^2s^2\right]ds\\ =&\frac{x^2(h-x)^2(2x-h)}{2}+\frac{h^3x(2h-3x)^3}{27(h-x)^2}.\end{aligned} \tag{4.10}$$

For $\frac{2h}{3}\leq x\leq h$, by a simple computation, it follows that

$$\int_x^h \left|h^3x-4h^2xs+3hxs^2+3hx^2s-3x^2s^2\right|ds=\frac{x^2(h-x)^2(2x-h)}{2}. \tag{4.11}$$

Combining (4.7)-(4.11), we obtain

$$\int_0^h |K_2(x,s)|ds=\begin{cases}\frac{x(2h-3x)^3}{27(h-x)^2}, & 0\leq x\leq \frac{h}{3};\\ \frac{x(2h-3x)^3}{27(h-x)^2}+\frac{(h-x)(3x-h)^3}{27x^2}, & \frac{h}{3}\leq x\leq\frac{2h}{3};\\ \frac{(h-x)(3x-h)^3}{27x^2}, & \frac{2h}{3}\leq x\leq h.\end{cases} \tag{4.12}$$

Similar to (3.20), by (4.12), we get

$$\max_{0\leq x\leq h}\int_0^h |K_2(x,s)|ds=\int_0^h\left|K_2\left(\frac{(5-\sqrt{13})h}{6},s\right)\right|ds=\frac{(13\sqrt{13}-46)h^2}{27}. \tag{4.13}$$

Similar to the proof of (1.2), by (4.6) and (4.13) we get (1.5).

For $p=1$, from (2.5) and (4.5), it follows that for $f\in W_1^3(\mathbb{R})$,

$$\int_0^h \left|H_n'(f,x)-f'(x)\right|dx \le \max_{0\le s\le h}\int_0^h |K_2(x,s)|dx\cdot\int_0^h \left|f^{(3)}(s)\right|ds. \tag{4.14}$$

From (4.4) we know $K_2(x,s)=K_2(s,x)$. Combining this fact with (4.13) we obtain

$$\max_{0\le s\le h}\int_0^h |K_2(x,s)|dx=\frac{(13\sqrt{13}-46)h^2}{27}. \tag{4.15}$$

Similar to the proof of (1.3), by (4.14) and (4.15) we get (1.6). By (2.2), (1.5) and (1.6) we get (1.7). □

5. Proof of Theorem 1.3

Proof. For $x\in[0,h]$, from $H_n(f,0)=f(0)$, Newton-Leibniz formula and (4.1)-(4.3), it follows that

$$\begin{aligned}
&H_n(f,x)-f(x)\\
=&\int_0^x \left[H_n'(f,t)-f'(t)\right]dt\\
=&\frac{1}{h^3}\int_0^x dt\int_0^t f^{(3)}(s)\left[h^3s-4h^2ts+3hts^2+3ht^2s-3t^2s^2\right]ds\\
&+\frac{1}{h^3}\int_0^x dt\int_t^h f^{(3)}(s)\left[h^3t-4h^2ts+3hts^2+3ht^2s-3t^2s^2\right]ds.
\end{aligned} \tag{5.1}$$

Exchanging the integral order, we obtain

$$\begin{aligned}
&\int_0^x dt\int_0^t f^{(3)}(s)\left[h^3s-4h^2ts+3hts^2+3ht^2s-3t^2s^2\right]ds\\
=&\int_0^x f^{(3)}(s)ds\int_s^x \left[h^3s-4h^2ts+3hts^2+3ht^2s-3t^2s^2\right]dt\\
=&\int_0^x f^{(3)}(s)\left[s^5-\frac{5hs^4}{2}+2h^2s^3+\left(\frac{3hx^2}{2}-h^3-x^3\right)s^2+hx(h-x)^2s\right]ds.
\end{aligned} \tag{5.2}$$

Similarly, one has

$$\begin{aligned}
&\int_0^x dt\int_t^h f^{(3)}(s)\left[h^3t-4h^2ts+3hts^2+3ht^2s-3t^2s^2\right]ds\\
=&\int_0^x dt\int_t^x f^{(3)}(s)\left[h^3t-4h^2ts+3hts^2+3ht^2s-3t^2s^2\right]ds\\
&+\int_0^x dt\int_x^h f^{(3)}(s)\left[h^3t-4h^2ts+3hts^2+3ht^2s-3t^2s^2\right]ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^x f^{(3)}(s)ds \int_0^s \left[h^3 t - 4h^2 ts + 3hts^2 + 3ht^2 s - 3t^2 s^2\right] dt \\
&\quad + \int_x^h f^{(3)}(s)ds \int_0^x \left[h^3 t - 4h^2 ts + 3hts^2 + 3ht^2 s - 3t^2 s^2\right] dt \\
&= \int_0^x f^{(3)}(s) \left[-s^5 + \frac{5hs^4}{2} - 2h^2 s^3 + \frac{h^3 s^2}{2}\right] ds \\
&\quad + \int_x^h f^{(3)}(s) \left[\frac{h^3 x^2}{2} + hx^2(x-2h)s + \frac{x^2(3h-2x)s^2}{2}\right] ds. \qquad (5.3)
\end{aligned}
$$

Denote

$$
K_3(x,s) = \begin{cases} \frac{\left(3hx^2 - h^3 - 2x^3\right)s^2 + 2hx(h-x)^2 s}{2h^3}, & 0 \le s \le x \le h; \\ \frac{h^3x^2 + 2hx^2(x-2h)s + x^2(3h-2x)s^2}{2h^3}, & 0 \le x \le s \le h. \end{cases} \qquad (5.4)
$$

By (5.1)-(5.4), we know

$$
H_n(f,x) - f(x) = \int_0^h f^{(3)}(s) K_3(x,s) ds. \qquad (5.5)
$$

For $p=\infty$, from (2.4) and (5.5), it follows that

$$
\sup_{f\in W^3_\infty(\mathbb{R})} \max_{0\le x\le h} |H_n(f,x) - f(x)| = \max_{0\le x\le h} \int_0^h |K_3(x,s)| ds. \qquad (5.6)
$$

From (5.4), it follows that for an arbitrary $0 \le x \le h$,

$$
\begin{aligned}
\int_0^h |K_3(x,s)| ds =& \frac{1}{2h^3} \int_0^x \left|\left(3hx^2 - h^3 - 2x^3\right) s^2 + 2hx(h-x)^2 s\right| ds \\
&+ \frac{1}{2h^3} \int_x^h \left|h^3x^2 + 2hx^2(x-2h)s + x^2(3h-2x)s^2\right| ds. \qquad (5.7)
\end{aligned}
$$

We consider the first integral in (5.7) now. For $0 \le x \le \frac{h}{2}$, it is easy to verify that

$$
\int_0^x \left|\left(3hx^2 - h^3 - 2x^3\right) s^2 + 2hx(h-x)^2 s\right| ds = \frac{2x^3(h-x)^3}{3}. \qquad (5.8)
$$

For $\frac{h}{2} \le x \le h$, by a direct computation, we obtain

$$
\begin{aligned}
&\int_0^x \left|\left(3hx^2 - h^3 - 2x^3\right) s^2 + 2hx(h-x)^2 s\right| ds \\
=& \int_0^{\frac{2hx}{2x+h}} \left[\left(3hx^2 - h^3 - 2x^3\right) s^2 + 2hx(h-x)^2 s\right] ds \\
&- \int_{\frac{2hx}{2x+h}}^x \left[\left(3hx^2 - h^3 - 2x^3\right) s^2 + 2hx(h-x)^2 s\right] ds \\
=& -\frac{2x^3(h-x)^3}{3} + \frac{8h^3x^3(h-x)^2}{3(2x+h)^2}. \qquad (5.9)
\end{aligned}
$$

We will consider the second integral in (5.7). For $0 \le x \le \frac{h}{2}$, it is easy to verify that

$$\begin{aligned}
&\int_x^h \left|h^3x^2 + 2hx^2(x-2h)s + x^2(3h-2x)s^2\right| ds \\
=&\int_x^{\frac{h^2}{3h-2x}} \left[h^3x^2 + 2hx^2(x-2h)s + x^2(3h-2x)s^2\right] ds \\
&-\int_{\frac{h^2}{3h-2x}}^h \left[h^3x^2 + 2hx^2(x-2h)s + x^2(3h-2x)s^2\right] ds \\
=&-\frac{2x^3(h-x)^3}{3} + \frac{8h^3x^2(h-x)^3}{3(3h-2x)^2}. \qquad (5.10)
\end{aligned}$$

For $\frac{h}{2} \le x \le h$, by a simple computation, it follows that

$$\int_x^h \left|h^3x^2 + 2hx^2(x-2h)s + x^2(3h-2x)s^2\right| ds = \frac{2x^3(h-x)^3}{3}. \qquad (5.11)$$

Combining (5.7)-(5.11), we obtain

$$\int_0^h |K_3(x,s)|ds = \begin{cases} \frac{4x^2(h-x)^3}{3(3h-2x)^2}, & 0 \le x \le \frac{h}{2}; \\ \frac{4x^3(h-x)^2}{3(2x+h)^2}, & \frac{h}{2} \le x \le h. \end{cases} \qquad (5.12)$$

Similar to (3.20), by (5.12), we get

$$\max_{0\le x\le h} \int_0^h |K_3(x,s)|ds = \int_0^h \left|K_3\left(\frac{h}{2}, s\right)\right| ds = \frac{h^3}{96}. \qquad (5.13)$$

Similar to the proof of (1.2), by (5.6) and (5.13), we get (1.8).

For $p=1$, from (2.5) and (5.5), it follows that for $f \in W_1^3(\mathbb{R})$,

$$\int_0^h |H_n(f,x) - f(x)|\, dx \le \max_{0\le s\le h} \int_0^h |K_3(x,s)|dx \cdot \int_0^h \left|f^{(3)}(s)\right| ds. \qquad (5.14)$$

From (5.4) we obtain

$$\begin{aligned}
\int_0^h |K_3(x,s)|dx =&\frac{1}{2h^3}\int_s^h \left|(2(h-s)x - hs)(h-x)^2 s\right| dx \\
&+ \frac{1}{2h^3}\int_0^s x^2(h-s)\left|2sx + h^2 - 3hs\right| dx. \qquad (5.15)
\end{aligned}$$

If $0 \le s \le \frac{h}{2}$, then

$$\int_s^h \left|(2(h-s)x - hs)(h-x)^2 s\right| dx = \int_s^h (2(h-s)x - hs)(h-x)^2 s dx. \qquad (5.16)$$

If $\frac{h}{2} \leq s \leq \frac{2h}{3}$, then

$$\int_s^h \left|(2(h-s)x-hs)(h-x)^2 s\right| dx = -\int_s^{\frac{sh}{2(h-s)}} (2(h-s)x-hs)(h-x)^2 s dx + \int_{\frac{sh}{2(h-s)}}^h (2(h-s)x-hs)(h-x)^2 s dx. \tag{5.17}$$

If $\frac{2h}{3} \leq s \leq h$, then

$$\int_s^h \left|(2(h-s)x-hs)(h-x)^2 s\right| dx = -\int_s^h (2(h-s)x-hs)(h-x)^2 s dx. \tag{5.18}$$

If $0 \leq s \leq \frac{h}{3}$, then

$$\int_0^s \left|x^2(h-s)(2sx+h^2-3hs)\right| dx = \int_0^s x^2(h-s)(2sx+h^2-3hs) dx. \tag{5.19}$$

If $\frac{h}{3} \leq s \leq \frac{h}{2}$, then

$$\int_0^s \left|x^2(h-s)(2sx+h^2-3hs)\right| dx = -\int_0^{\frac{3hs-h^2}{2s}} x^2(h-s)(2sx+h^2-3hs) dx + \int_{\frac{3hs-h^2}{2s}}^s x^2(h-s)(2sx+h^2-3hs) dx. \tag{5.20}$$

If $\frac{h}{2} \leq s \leq h$, then

$$\int_0^s \left|x^2(h-s)(2sx+h^2-3hs)\right| dx = -\int_0^s x^2(h-s)(2sx+h^2-3hs) dx. \tag{5.21}$$

From (5.15)-(5.21) and a direct computation, we obtain

$$\int_0^h |K_3(x,s)| dx = \begin{cases} \frac{s(h-s)(h-2s)}{12}, & 0 \leq s \leq \frac{h}{3}; \\ \frac{s(h-s)(h-2s)}{12} + \frac{h(h-s)(3s-h)^4}{96s^3}, & \frac{h}{3} \leq s \leq \frac{h}{2}; \\ \frac{s(h-s)(2s-h)}{12} + \frac{hs(2h-3s)^4}{96(h-s)^3}, & \frac{h}{2} \leq s \leq \frac{2h}{3}; \\ \frac{s(h-s)(2s-h)}{12}, & \frac{2h}{3} \leq s \leq h. \end{cases} \tag{5.22}$$

Similar to (3.20), by (5.22) we obtain that

$$\max_{0 \leq s \leq h} \int_0^h |K_3(x,s)| dx = \int_0^h \left| K_3\left(x, \frac{3-\sqrt{3}}{6} h\right)\right| dx = \frac{\sqrt{3}}{216} h^3. \tag{5.23}$$

Similar to the proof of (1.3), by (5.14) and (5.23) we get (1.9). From (2.2), (1.8) and (1.9) we obtain (1.10). □

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