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Difference Approximation of Stochastic Elastic Equation Driven by Infinite Dimensional Noise

Yinghan Zhang*, Xiaoyuan Yang and Ruisheng Qi

Department of Mathematics, Beihang University, LMIB of the Ministry of Education, Beijing 100191, China.

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Abstract. An explicit difference scheme is described, analyzed and tested for numerically approximating stochastic elastic equation driven by infinite dimensional noise. The noise processes are approximated by piecewise constant random processes and the integral formula of the stochastic elastic equation is approximated by a truncated series. Error analysis of the numerical method yields estimate of convergence rate. The rate of convergence is demonstrated with numerical experiments.

AMS subject classifications: 60H15; 65M06

Key words: Stochastic partial differential equations, difference scheme, stochastic elastic equation, infinite dimensional noise, rate of convergence.

1. Introduction

The subject of stochastic partial differential equations (SPDEs) has gained considerable popularity and importance due to its frequent appearance in various fields, such as mechanics, biology, chemistry, epidemiology, microelectronics, economics and finance. SPDEs can describe many phenomena in various fields of science and engineering. During the past decades, there has been an increasing demand for tools and methods of SPDEs in various disciplines and many theoretical analyses for SPDEs have been studied theoretically, for example [9,12,18,24,25,28,30]. The numerical analysis of SPDEs is a young topic of research. Recently, many useful numerical methods for SPDEs have been developed, for instance, finite differences [1–3,6–14,16,19,23,26,27,29,32,33], finite elements [4,5,21,31].

For the contributions on numerical approximating parabolic SPDEs, we refer the reader to [1,2,7,8,10,13,14,16,19,26,29,32,33] and reference therein. In [1] the finite element and difference methods were studied for some linear SPDEs. I. Gyöngy and D. Nualart [13] introduced an implicit numerical scheme for a stochastic parabolic equation and showed that it converges uniformly in probability. I. Gyöngy [14] also

*Corresponding author. *Email address:* zhangyinghan007@126.com (Y. -H. Zhang)

applied finite difference to stochastic parabolic equations and derived the rate of convergence in L^p. In [15], a finite difference approximation scheme for an elliptic SPDE in dimension $d(d = 1, 2, 3)$ was studied and estimates for the rate of convergence of the approximations were obtained. In [8], the authors studied finite element approximations of some linear parabolic and elliptic SPDEs driven by special additive noises. The effects of the noise on the accuracy of the approximation were discussed. Annie Millet and Pierre-Luc Morine [22] studied the speed of convergence of the explicit and implicit space-time discretization schemes of the solution $u(t, x)$ to a parabolic partial differential equation in any dimension perturbed by a space-correlated Gaussian noise. The influence of the correlation on the speed was observed. For the numerical approximating of hyperbolic SPDEs, we refer the reader to [23, 27] and the reference therein.

It should be noted that most of the papers on numerically approximating SPDEs by finite difference approximation are devoted to the case of space-time white noise. However, there are few papers deal with the SPDEs driven by infinite dimensional noise by a finite difference method.

Enlightened by the above contributions, in this paper we consider strong approximations for a stochastic elastic equation in spatial dimension $d = 1, 2, \text{or} 3$ by an explicit difference scheme. To our best knowledge, this is a first step towards the analysis of lattice approximations for stochastic elastic equation driven by infinite dimensional noise.

Let $D = [0, 1]^d, (d = 1, 2, \text{ or } 3)$, consider the numerical approximation of the following stochastic partial differential equation

$$\begin{cases} u_{tt} + \Delta^2 u = f(t, x, u) + \dot{W}(t), \ t \geq 0, x \in D, \\ u = \Delta u = 0 \text{ on } \partial D, \\ u|_{t=0} = u_0, \ u_t|_{t=0} = v_0, \text{ on } D, \end{cases} \tag{1.1}$$

where $W(t)$ is a Hilbert space $U = L^2(D)$ valued Q-Wiener process defined as follows. Q is a symmetric bounded nonnegative operator on $L^2(D)$, there exists a complete orthonormal system $\{e_k\}$ in $L^2(D)$ and a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k, k = 1, 2, \ldots$, then W has the expansion

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k. \tag{1.2}$$

And so

$$\dot{W}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \dot{\beta}_k(t) e_k, \tag{1.3}$$

where $\beta_k(t), (k = 1, 2, \ldots)$ are real valued Brownian notions mutually independent on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\dot{\beta}_k(t)$ is the derivative of $\beta_k(t)$. If $TrQ = \Sigma_{k=1}^{\infty} \lambda_k < +\infty$, then the series (1.2) is convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$ and $\dot{W}(t)$ is called colored noise. If $TrQ = +\infty$, then the series (1.2) is not convergent in U, but convergent in a suitable Hilbert space U_1 such that U is embedded continuously into U_1 and

$\dot{W}(t)$ is called white noise. For more properties about the Hilbert space valued Wiener process, we refer the reader to [24].

Stochastic elastic equation is a fourth order partial differential equation and has very wide applications in structural engineering. As an engineering problem, it has its applications in beams, bridges and other structures, see [3, 24].

Throughout the paper, we use the letter C denotes a constant that may not be the same form one occurrence to anther, even in the same line. We express the dependence on some parameters by writing the parameters as arguments, e.g., $C = C(\alpha)$.

The rest of the paper is organized as follows. In the next section, we study the Hölder continuity of the sample paths of the solution of Eq. (1.1). Section 3 is devoted to approximate the noise and the integral formulation. In Section 4, we give the explicit difference scheme and prove the rate of convergence. Finally in Section 5, numerical experiments demonstrate the theoretical analysis.

2. Regularity of the solution

In this section, we present the conditions under which Eq. (1.1) has a unique solution and prove boundedness and Hölder continuity of the sample paths of the solution.

We fix a finite time horizon T and assume that the nonlinear term f satisfies the following Lipschitz and linear growth conditions

$$|f(t,x,z_1) - f(s,y,z_2)| \le C(|t-s| + |x-y| + |z_1 - z_2|), \tag{2.1}$$

$$\sup_{(t,x)\in[0,T]\times[0,1]} |f(t,x,z)| \le C(1+|z|), \tag{2.2}$$

for every $s,t \in [0,T], z_1, z_2, z \in \mathbb{R}, x = (x_1,\dots,x_d), y = (y_1,\dots,y_d) \in [0,1]^d$, $|x| = (\sum_{i=1}^d x_i^2)^{\frac{1}{2}}$, and C is a positive constant.

Let $r = (r_1,\dots,r_d) \in \mathbb{N}^d, x = (x_1,\dots,x_d) \in \mathbb{R}^d$. Define $\phi_r(x) = \sqrt{2^d}\prod_{i=1}^d \sin(r_i \pi x_i)$. Then $\{\phi_r\}_{r\in\mathbb{N}^d}$ satisfy the boundary conditions of (1.1) and compose of a complete orthonormal system on $L^2(D)$ which diagonalize Δ with

$$\gamma_r = \pi^2|r|^2 = \pi^2 \sum_{i=1}^d r_i^2, \tag{2.3}$$

the corresponding eigenvalues. We assume that the complete orthonormal system $\{\phi_r\}_{r\in\mathbb{N}^d}$ diagonalize Q with corresponding eigenvalues $\{\lambda_r\}_{r\in\mathbb{N}^d}$. Since Q is bounded, $\{\lambda_r\}_{r\in\mathbb{N}^d}$ is a bounded sequence too. Therefore, in the following we always assume that $\dot{W}$ has formal expression

$$\dot{W}(t) = \sum_{r\in\mathbb{N}^d} \sqrt{\lambda_r}\dot{\beta}_r(t)\phi_r, \tag{2.4}$$

and the degenerate rate of λ_r is

$$\lambda_r = O(|r|^{-\lambda}), \lambda \ge 0. \tag{2.5}$$

Assume that u_0, v_0 belong to $L^2(D)$. Since the fundamental solution of

$$\begin{aligned}&v_{tt}+\Delta^2 v=0,\ t\geq 0, x\in D,\\&v=\Delta u=0 \text{ on } \partial D,\\&v|_{t=0}=\phi(x),\ v_t|_{t=0}=\psi(x), x\in D,\end{aligned}$$

is given by

$$v(t,x)=\sum_{r\in\mathbb{N}^d}\frac{\sin(\gamma_r t)}{\gamma_r}\varphi_r(x)\int_D\psi(y)\varphi_r(y)\mathrm{d}y+\sum_{r\in\mathbb{N}^d}\cos(\gamma_r t)\varphi_r(x)\int_D\phi(y)\varphi_r(y)\mathrm{d}y.$$

From the theoretical results for stochastic elastic equation discussed in [3,6,11,17,33], we know that under the conditions (2.1) and (2.2), problem (1.1) has a unique weak solution and is given by the the integral formulation

$$\begin{aligned}u(t,x) \;=\;& \sum_{r\in\mathbb{N}^d}\frac{\sin(\gamma_r t)}{\gamma_r}\phi_r(x)\int_D v_0(y)\phi_r(y)\mathrm{d}y\\&+\sum_{r\in\mathbb{N}^d}\cos(\gamma_r t)\phi_r(x)\int_D u_0(y)\phi_r(y)\mathrm{d}y\\&+\sum_{r\in\mathbb{N}^d}\int_0^t\int_D\frac{\sin(\gamma_r(t-s))}{\gamma_r}\phi_r(x)\phi_r(y)f(s,y,u(s,y))\mathrm{d}s\mathrm{d}y\\&+\sum_{r\in\mathbb{N}^d}\int_0^t\int_D\frac{\sin(\gamma_r(t-s))}{\gamma_r}\phi_r(x)\phi_r(y)\dot{W}(s,y)\mathrm{d}s\mathrm{d}y. \end{aligned}\tag{2.6}$$

For a given function $g: D\to\mathbb{R}$ and $\alpha\in\mathbb{R}$, define

$$\|g\|_{\alpha,2}:=\Big(\sum_{r\in\mathbb{N}^d}(1+|r|^2)^\alpha|<g,\varphi_r>|^2\Big)^{\frac{1}{2}},$$

where $<\cdot,\cdot>$ stands for the usual scalar product in $L^2(D)$, and denote by $H^{\alpha,2}(D)$ the set of functions $g: D\to\mathbb{R}$ such that $\|g\|_{\alpha,2}<\infty$. Notice that $H^{\alpha,2}(D)$ is a subspace of the fractional Sobolev space of fractional differential order α and integrability order $p=2$ (see [27]). For a special case $\alpha=0$, it is clear that $H^{\alpha,2}(D)=L^2(D)$ and we will denote $\|\cdot\|_{0,2}$ by $\|\cdot\|$. By Parseval's identity, we have

$$\|g\|^2=\sum_{r\in\mathbb{N}^d}|<g,\varphi_r>|^2,\quad \forall g\in L^2(D). \tag{2.7}$$

The following lemma gives an asymptotic bounds for series that will often appear in the rest of the paper and for a proof of them we refer the reader to [20].

Lemma 2.1. *Let $d\in\{1,2,3\}$ and $c^*>0$. Then there exists a constant $C=C(c^*,d)>0$, such that*

$$\sum_{\alpha\in\mathbb{N}^d}\|\alpha\|_{\mathbb{N}^d}^{-d+c^*\varepsilon}\leq C\varepsilon^{-1},\forall\varepsilon\in(0,2].$$

The boundedness of the sample paths of the solution is given below.

Lemma 2.2. *Assume that* $v_0 \in H^{\beta,2}(D)$ *for some* $\beta > -\frac{4-d}{2}$ *and* $u_0 \in H^{\alpha,2}(D)$ *for some* $\alpha > \frac{d}{2}$, *and* f *satisfies the linear growth condition* (2.2). *Then, there exists a constant* $C = C(\alpha,\beta,\lambda,T)$, *such that*

$$\sup_{(t,x)\in[0,T]\times D} \mathbb{E}|u(t,x)|^2 < C.$$

Proof. By the expression (2.6), it is easy to see that

$$\mathbb{E}|u(t,x)|^2 \le 4\sum_{i=1}^{4} I_k(t,x),$$

with

$$I_1(t,x) = \left|\sum_{r\in\mathbb{N}^d} \frac{\sin(\gamma_r t)}{\gamma_r}\phi_r(x)\int_D v_0(y)\phi_r(y)\mathrm{d}y\right|^2,$$

$$I_2(t,x) = \left|\sum_{r\in\mathbb{N}^d} \cos(\gamma_r t)\phi_r(x)\int_D u_0(y)\phi_r(y)\mathrm{d}y\right|^2,$$

$$I_3(t,x) = \mathbb{E}\left|\sum_{r\in\mathbb{N}^d}\int_0^t\int_D \frac{\sin(\gamma_r(t-s))}{\gamma_r}\phi_r(x)\phi_r(y)f(s,y,u(s,y))\mathrm{d}s\mathrm{d}y\right|^2,$$

$$I_4(t,x) = \mathbb{E}\left|\sum_{r\in\mathbb{N}^d}\int_0^t\int_D \frac{\sin(\gamma_r(t-s))}{\gamma_r}\phi_r(x)\phi_r(y)\dot{W}(s,y)\mathrm{d}s\mathrm{d}y\right|^2.$$

By Lemma 2.1, Cauchy-Schwartz inequality and the assumptions on v_0, we have

$$\begin{aligned} I_1(t,x) &= \left|\sum_{r\in\mathbb{N}^d}\frac{\sin(\gamma_r t)}{\gamma_r}|r|^{-\beta}\phi_r(x) < v_0,\phi_r > |r|^{\beta}\right|^2 \\ &\le C\sum_{r\in\mathbb{N}^d} | < v_0,\phi_r > |^2|r|^{2\beta}\sum_{r\in\mathbb{N}^d}|r|^{-4-2\beta} \\ &\le C\|v_0\|^2_{\beta,2}\sum_{r\in\mathbb{N}^d}|r|^{-4-2\beta} \le C, \end{aligned}$$

if $\beta > -\frac{4-d}{2}$. Similarly, one has

$$I_2(t,x) \le C\|u_0\|^2_{\alpha,2}\sum_{r\in\mathbb{N}^d}|r|^{-2\beta} \le C,$$

if $\alpha > \frac{d}{2}$. Applying Cauchy-Schwartz inequality and the linear growth condition (2.2) yield

$$
\begin{aligned}
I_3(t,x) &\leq \mathbb{E}\left|\sum_{r\in\mathbb{N}^d}\int_0^t \frac{\sin(\gamma_r(t-s))\phi_r(x)}{\gamma_r}\left[\int_D \phi_r(y)f(s,y,u(s,y))\mathrm{d}y\right]\mathrm{d}s\right|^2 \\
&\leq \mathbb{E}\sum_{r\in\mathbb{N}^d}\int_0^t \left|\frac{\sin(\gamma_r(t-s))\phi_r(x)}{\gamma_r}\right|^2 \mathrm{d}s \int_0^t \sum_{r\in\mathbb{N}^d} |<\phi_r, f>|^2 \mathrm{d}s \\
&\leq C\sum_{r\in\mathbb{N}^d}\frac{1}{|r|^4}\int_0^t \mathbb{E}\|f(s,\cdot,u(s,\cdot))\|^2\mathrm{d}s \\
&\leq C\int_0^t 1+\mathbb{E}\|u(s,\cdot)\|^2\mathrm{d}s \\
&\leq C\left(1+\int_0^t \sup_{x\in D}\mathbb{E}|u(s,x)|^2\mathrm{d}s\right).
\end{aligned}
$$

By the white noise expression (2.4), the orthogonal property of ϕ_r and the mutual independence of β_r, one gets

$$
\begin{aligned}
I_4(t,x) &= \mathbb{E}\left|\sum_{r\in\mathbb{N}^d}\int_0^t\int_D \frac{\sin(\gamma_r(t-s))}{\gamma_r}\phi_r(x)\phi_r(y)\sum_{k\in\mathbb{N}^d}\sqrt{\lambda_k}\dot{\beta}_k(s)\phi_k(y)\mathrm{d}s\mathrm{d}y\right|^2 \\
&= \mathbb{E}\left|\sum_{r\in\mathbb{N}^d}\int_0^t \frac{\sin(\gamma_r(t-s))}{\gamma_r}\phi_r(x)\sqrt{\lambda_r}\mathrm{d}\beta_r(s)\right|^2 \\
&= \sum_{r\in\mathbb{N}^d}\int_0^t \frac{\sin^2(\gamma_r(t-s))}{\gamma_r^2}\phi_r^2(x)\lambda_r\mathrm{d}s \leq C\sum_{r\in\mathbb{N}^d}\frac{\lambda_r}{|r|^4}\leq C.
\end{aligned}
$$

Combining together the above estimates, we can obtain that

$$
\sup_{x\in D}\mathbb{E}(|u(t,x)|^2) \leq C\left(1+\int_0^t \sup_{x\in D}\mathbb{E}(|u(s,x)|^2)\mathrm{d}s\right),
$$

with a constant C independent of t. We conclude applying Gronwall's lemma. □

We next prove the Hölder property of the sample paths of the solution.

Lemma 2.3. *Assume that $v_0 \in H^{\beta,2}(D)$ for some $\beta > -\frac{4-d}{2}$ and $u_0 \in H^{\alpha,2}(D)$ for some $\alpha > \frac{d}{2}$, and f satisfies conditions (2.1) and (2.2). Then there exists a constant $C = C(\alpha,\beta,\lambda,T)$, such that, for every $s,t \in [0,T]$ and $x,y \in D$.*

$$
\begin{aligned}
&\mathbb{E}|u(t,x)-u(s,y)|^2 \\
&\leq C\left(|t-s|^{(2+\beta-\frac{d}{2})\wedge(\alpha-\frac{d}{2})\wedge(2-\frac{d}{2})} + |x-y|^{(4+2\beta-d)\wedge(2\alpha-d)\wedge(4-d)\wedge 2}\right),
\end{aligned}
$$

where $a \wedge b$ *stands for* $\min\{a, b\}$. *Moreover, if the nonlinear term* $f = 0$, *then for every* $s, t \in [0, T]$ *and* $x, y \in D$,

$$\begin{aligned}&\mathbb{E}|u(t,x)-u(s,y)|^2\\ \leq & C\left(|t-s|^{(2+\beta-\frac{d}{2})\wedge(\alpha-\frac{d}{2})\wedge(2-\frac{d}{2}+\frac{\lambda}{2})\wedge 2}+|x-y|^{(4+2\beta-d)\wedge(2\alpha-d)\wedge(4-d+\lambda)\wedge 2}\right).\end{aligned}$$

Proof. We assume that $0 \leq s < t \leq T$. Let

$$\begin{aligned}H(t,x)&=\sum_{r\in\mathbb{N}^d}\int_0^t\int_D\frac{\sin(\gamma_r(t-s))}{\gamma_r}\phi_r(x)\phi_r(z)f(s,z,u(s,z))\mathrm{d}s\mathrm{d}z,\\ F(t,x)&=\sum_{r\in\mathbb{N}^d}\int_0^t\int_D\frac{\sin(\gamma_r(t-s))}{\gamma_r}\phi_r(x)\phi_r(z)\dot{W}(s,z)\mathrm{d}s\mathrm{d}z.\end{aligned}$$

Then we have the following decomposition

$$\mathbb{E}|u(t,x)-u(s,y)|^2\leq 4\sum_{k=1}^4 J_k(t,x;s,y),$$

with

$$\begin{aligned}J_1(t,x;s,y)&=\left|\sum_{r\in\mathbb{N}^d}\left(\frac{\sin(\gamma_r t)}{\gamma_r}\phi_r(x)-\frac{\sin(\gamma_r s)}{\gamma_r}\phi_r(y)\right)<\phi_r,v_0>\right|^2,\\ J_2(t,x;s,y)&=\left|\sum_{r\in\mathbb{N}^d}(\cos(\gamma_r t)\phi_r(x)-\cos(\gamma_r s)\phi_r(y))<\phi_r,u_0>\right|^2,\\ J_3(t,x;s,y)&=\mathbb{E}\left|H(t,x)-H(s,y)\right|^2,\quad J_4(t,x;s,y)=\mathbb{E}\left|F(t,x)-F(s,y)\right|^2.\end{aligned}$$

Cauchy-Schwartz inequality and the assumptions on v_0 yield

$$\begin{aligned}&J_1(s,x;t,y)\leq\|v_0\|^2_{\beta,2}\sum_{r\in\mathbb{N}^d}\left|\frac{\sin(\gamma_r t)\phi_r(x)-\sin(\gamma_r s)\phi_r(y)}{\gamma_r}\right|^2|r|^{-2\beta}\\ &\leq C\left(\sum_{r\in\mathbb{N}^d}\left|\frac{\sin(\gamma_r t)-\sin(\gamma_r s)}{\gamma_r}\phi_r(x)\right|^2|r|^{-2\beta}\right.\\ &\qquad\left.+\sum_{r\in\mathbb{N}^d}\left|\frac{\sin(\gamma_r t)}{\gamma_r}(\phi_r(x)-\phi_r(y))\right|^2|r|^{-2\beta}\right)\\ &=:C(J_{11}+J_{12}).\end{aligned}$$

The mean value theorem yields

$$J_{11}\leq C\sum_{r\in\mathbb{N}^d}|r|^{-4-2\beta}\left(1\wedge|r|^4(t-s)^2\right).$$

If $\beta > \frac{d}{2}$, it is clear that $J_{11} \leq C(t-s)^2$. If $-\frac{4-d}{2} < \beta \leq \frac{d}{2}$,

$$\begin{aligned} J_{11} &\leq C \sum_{|r| \leq [(t-s)^{-\frac{1}{2}}]} |r|^{-2\beta}(t-s)^2 + \sum_{|r| > [(t-s)^{-\frac{1}{2}}]} |r|^{-4-2\beta} \\ &\leq C((t-s)^2(t-s)^{\beta-\frac{d}{2}} + (t-s)^{2+\beta-\frac{d}{2}} \\ &\leq C(t-s)^{2+\beta-\frac{d}{2}}, \end{aligned}$$

where $[\cdot]$ stands for the integer value. The above arguments implies that

$$J_{11} \leq C(t-s)^{(2+\beta-\frac{d}{2})\wedge 2}.$$

Analogously, $J_{12} \leq C|x-y|^{(4+2\beta-d)\wedge 2}$. Thus

$$J_1(s,x;t,y) \leq C\Big(|t-s|^{(2+\beta-\frac{d}{2})\wedge 2} + |x-y|^{(4+2\beta-d)\wedge 2}\Big). \tag{2.8}$$

Now we deal with the term J_2. By Cauchy-Schwartz inequality and the assumptions on u_0,

$$\begin{aligned} J_2(s,x;t,y) &\leq \|u_0\|^2_{\alpha,2} \sum_{r\in\mathbb{N}^d} \Big|\cos(\gamma_r t)\phi_r(x) - \cos(\gamma_r s)\phi_r(y)\Big|^2 |r|^{-2\alpha} \\ &\leq C\left(\sum_{r\in\mathbb{N}^d} \Big|\cos(\gamma_r t) - \cos(\gamma_r s)|^2 |r|^{-2\alpha} + \sum_{r\in\mathbb{N}^d} |\phi_r(x) - \phi_r(y)\Big|^2 |r|^{-2\alpha}\right). \end{aligned}$$

Similar to the method used in the analysis of the term J_1, one has

$$J_2(s,x;t,y) \leq C\Big(|t-s|^{(\alpha-\frac{d}{2})\wedge 2} + |x-y|^{(2\alpha-d)\wedge 2}\Big). \tag{2.9}$$

To give a estimation of J_3, we consider the decomposition

$$\begin{aligned} J_3(s,x;t,y) &= \mathbb{E}\,|H(t,x) - H(s,y)|^2, \\ &\leq C\left(\mathbb{E}\,|H(t,x) - H(s,x)|^2 + \mathbb{E}\,|H(s,x) - H(s,y)|^2\right) \\ &=: C(J_{31} + J_{32}). \end{aligned}$$

By the linear growth condition (2.2) and lemma 2.2, one gets

$$
\begin{aligned}
J_{31} =& \mathbb{E}\Bigg|\sum_{r\in\mathbb{N}^d}\int_0^t\int_D \frac{\sin(\gamma_r(t-\tau))}{\gamma_r}\phi_r(x)\phi_r(z)f(\tau,z,u(\tau,z))\mathrm{d}\tau\mathrm{d}z \\
&\quad - \sum_{r\in\mathbb{N}^d}\int_0^s\int_D \frac{\sin(\gamma_r(s-\tau))}{\gamma_r}\phi_r(x)\phi_r(z)f(\tau,z,u(\tau,z))\mathrm{d}\tau\mathrm{d}z\Bigg|^2 \\
\le& C\mathbb{E}\Bigg|\sum_{r\in\mathbb{N}^d}\int_0^s\int_D \frac{\sin(\gamma_r(t-\tau))-\sin(\gamma_r(s-\tau))}{\gamma_r}\phi_r(x)\phi_r(z)f(\tau,z,u)\mathrm{d}\tau\mathrm{d}z \\
&\quad + \sum_{r\in\mathbb{N}^d}\int_s^t\int_D \frac{\sin(\gamma_r(t-\tau))}{\gamma_r}\phi_r(x)\phi_r(z)f(\tau,z,u(\tau,z))\mathrm{d}\tau\mathrm{d}z\Bigg|^2 \\
=:& J_{311}+J_{312}.
\end{aligned}
$$

By (2.2), Lemma 2.2 and Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
J_{311} &\le C\sum_{r\in\mathbb{N}^d}\frac{1\wedge|r|^4|t-s|^2}{|r|^4}\int_0^s(1+\sup_{x\in D}\mathbb{E}|u(s,x)|^2)\mathrm{d}\tau \\
&\le C\left(\sum_{|r|\le[|t-s|^{-\frac12}]}|t-s|^2+\sum_{|r|>[|t-s|^{-\frac12}]}|r|^{-4}\right) \\
&\le C|t-s|^{2-\frac{d}{2}}.
\end{aligned}
$$

Similarly, we also have

$$J_{312}\le C|t-s|^{2-\frac{d}{2}}.$$

Apply the linear growth condition (2.2) and Lemma 2.2 and then the orthogonal property of ϕ_r, we have

$$
\begin{aligned}
J_{32} &= \mathbb{E}\left|\sum_{r\in\mathbb{N}^d}\int_0^s\int_D\frac{\sin(\gamma_r(s-\tau))}{\gamma_r}(\phi_r(x)-\phi_r(y))\phi_r(z)f(\tau,z,u)\mathrm{d}\tau\mathrm{d}z\right|^2 \\
&\le C\sum_{r\in\mathbb{N}^d}\frac{1\wedge|r|^2|x-y|^2}{|r|^4}\le C|x-y|^{(4-d)\wedge 2}.
\end{aligned}
$$

Combining together the estimations of J_{31} and J_{32}, we obtain that

$$J_3(s,x;t,y)\le C\left(|t-s|^{2-\frac{d}{2}}+|x-y|^{(4-d)\wedge 2}\right). \tag{2.10}$$

The white noise expression (2.4) and the orthogonal property of $\phi_r(x)$ yield

$$
\begin{aligned}
J_4(s,x;t,y) =& \mathbb{E}\left|\sum_{r\in\mathbb{N}^d}\left(\int_0^t \frac{\sin(\gamma_r(t-\tau))}{\gamma_r}\phi_r(x)\sqrt{\lambda_r}\mathrm{d}\beta_r(\tau)\right.\right. \\
&\left.\left. - \int_0^s \frac{\sin(\gamma_r(s-\tau))}{\gamma_r}\phi_r(y)\sqrt{\lambda_r}\mathrm{d}\beta_r(\tau)\right)\right|^2 \\
\le& C\mathbb{E}\left|\sum_{r\in\mathbb{N}^d}\int_0^s \frac{\sin(\gamma_r(t-\tau))-\sin(\gamma_r(s-\tau))}{\gamma_r}\phi_r(x)\sqrt{\lambda_r}\mathrm{d}\beta_r(\tau)\right|^2 \\
&+C\mathbb{E}\left|\sum_{r\in\mathbb{N}^d}\int_0^s \frac{\sin(\gamma_r(s-\tau))}{\gamma_r}(\phi_r(x)-\phi_r(y))\sqrt{\lambda_r}\mathrm{d}\beta_r(\tau)\right|^2 \\
&C\mathbb{E}\left|\sum_{r\in\mathbb{N}^d}\int_s^t \frac{\sin(\gamma_r(t-\tau))}{\gamma_r}\phi_r(x)\sqrt{\lambda_r}\mathrm{d}\beta_r(\tau)\right|^2 \\
=:& J_{41}+J_{42}+J_{43}.
\end{aligned}
$$

By using the mutual independence of $\beta_r(t)$, the orthogonal property of $\phi_r(x)$ and the method that used in the estimate of J_1, we have

$$
\begin{aligned}
&J_{41} \le C\sum_{r\in\mathbb{N}^d}\frac{1\wedge|r|^4|t-s|^2}{|r|^4}\lambda_r \le C|r|^{-4-\lambda}\left(1\wedge|r|^4|t-s|^2\right) \le C|t-s|^{(2-\frac{d}{2}+\frac{\lambda}{2})\wedge 2}, \\
&J_{42} \le C\sum_{r\in\mathbb{N}^d}\frac{1\wedge|r|^2|x-y|^2}{|r|^4}\lambda_r \le C|r|^{-4-\lambda}\left(1\wedge|r|^2|x-y|^2\right) \le C|x-y|^{(4-d+\lambda)\wedge 2}, \\
&J_{43} \le C|t-s|^{(2-\frac{d}{2}+\frac{\lambda}{2})\wedge 2}.
\end{aligned}
$$

Thus

$$
J_4(s,x;t,y) \le C\left(|t-s|^{(2-\frac{d}{2}+\frac{\lambda}{2})\wedge 2}+|x-y|^{(4-d+\lambda)\wedge 2}\right). \tag{2.11}
$$

Combining together the estimate of $J_i, (i=1,2,3,4)$, we finished the proof. □

3. White noise and integral formula approximation

In this section, we regularize the noise through time discretization and approximate the integral formula (2.6) by a truncated series.

First we regularize the noise by the following manner. Define a partition of $[0,T]$ by $[t_i,t_{i+1}]$ for $i=0,1,2,\ldots,m-1$, where $t_i=i\Delta t$, $\Delta t=\frac{T}{m}$. A sequence of noise which approximates W is defined as

$$
\dot{W}_m(t,x) = \sum_{r\in\mathbb{N}^d}\sqrt{\lambda_r}\phi_r(x)\sum_{i=1}^m \frac{1}{\sqrt{\Delta t}}\beta_{ri}\chi_i(t), \tag{3.1}
$$

where $\chi_i(t)$ is the characteristic function for the ith time subinterval and

$$\beta_{ri} = \frac{1}{\sqrt{\Delta t}} \int_{t_{i-1}}^{t_i} \mathrm{d}\beta_r(t), \tag{3.2}$$

that is to say, β_{ri} are mutually independent and $\beta_{ri} \in N(0,1)$.

Replacing $\dot{W}(t,x)$ in (2.4) by $\dot{W}_m(t,x)$ and let u_m be the solution of the following equation

$$\begin{aligned} u_m(t,x) =& \sum_{r\in\mathbb{N}^d} \frac{\sin(\gamma_r t)}{\lambda_r}\phi_r(x) \int_D v_0(y)\phi_r(y)\mathrm{d}y \\ &+ \sum_{r\in\mathbb{N}^d} \cos(\gamma_r t)\phi_r(x) \int_D u_0(y)\phi_r(y)\mathrm{d}y \\ &+ \sum_{r\in\mathbb{N}^d} \int_0^t \int_D \frac{\sin(\gamma_r(t-s))}{\gamma_r}\phi_r(x)\phi_r(y) f(s,y,u_m(s,y))\mathrm{d}s\mathrm{d}y \\ &+ \sum_{r\in\mathbb{N}^d} \int_0^t \int_D \frac{\sin(\gamma_r(t-s))}{\gamma_r}\phi_r(x)\phi_r(y)\dot{W}_m(s,y)\mathrm{d}s\mathrm{d}y. \end{aligned} \tag{3.3}$$

The next theorem shows that under suitable conditions, u_m approximates u, the solution of (1.1).

Theorem 3.1. *Suppose that the Lipschitz condition* (2.1) *is satisfied, then there exists a constant $C = C(\lambda, T)$, such that*

$$\sup_{t\in[0,T]} \mathbb{E}\left| \int_D |u(t,x) - u_m(t,x)|^2 \mathrm{d}x \right| \le C\Delta t^{(2-\frac{d}{2}+\frac{\lambda}{2})\wedge 2}.$$

Proof. Let $e(t,x) = u(t,x) - u_m(t,x)$, and

$$\begin{aligned} A(t,x) &= \sum_{r\in\mathbb{N}^d} \int_0^t \int_D \frac{\sin(\gamma_r(t-s))\phi_r(x)\phi_r(y)}{\gamma_r} \left(f(s,y,u) - f(s,y,u_m)\right)\mathrm{d}s\mathrm{d}y, \\ B(t,x) &= \sum_{r\in\mathbb{N}^d} \int_0^t \int_D \frac{\sin(\gamma_r(t-s))\phi_r(x)\phi_r(y)}{\gamma_r} \left(\dot{W}(s,y) - \dot{W}_m(s,y)\right)\mathrm{d}s\mathrm{d}y. \end{aligned}$$

Then, we have $e(t,x) = A(t,x) + B(t,x)$ and $e^2(t,x) \le 2(A^2(t,x) + B^2(t,x))$. By Cauchy-Schwartz inequality, Lemma 2.1 and the Lipschitz condition (2.1), one has

$$\begin{aligned} A^2(t,x) &\le C \sum_{r\in\mathbb{N}^d} \int_0^t \frac{\sin^2(\gamma_r(t-s))}{\gamma_r^2}\mathrm{d}s \int_0^t \sum_{r\in\mathbb{N}^d} |< u-u_m, \phi_r >|^2 \mathrm{d}s \\ &\le C\int_0^t \|e(s,\cdot)\|^2 \mathrm{d}s. \end{aligned}$$

Now we estimate $B(t,x)$. For simplicity, we assume $t=t_{k+1}, 0\le k<m$, by the white noise approximation (3.1) and the orthogonal property of $\phi_r(y)$, one gets

$$
\begin{aligned}
B(t,x) &= \sum_{r\in\mathbb{N}^d}\int_0^t\int_D \frac{\sin(\gamma_r(t-s))\phi_r(x)\phi_r(y)}{\gamma_r} \\
&\quad\times\sum_{l\in\mathbb{N}^d}\sqrt{\lambda_l}\phi_l(y)\left[\dot\beta_l(s)-\sum_{i=1}^m\frac{1}{\Delta t}\int_{t_{i-1}}^{t_i}\mathrm{d}\beta_l(\tau)\right]\mathrm{d}s\mathrm{d}y \\
&=\sum_{r\in\mathbb{N}^d}\int_0^t\frac{\sin(\gamma_r(t-s))\phi_r(x)}{\gamma_r}\sqrt{\lambda_r}\left[\dot\beta_l(s)-\sum_{i=1}^m\frac{1}{\Delta t}\int_{t_{i-1}}^{t_i}\mathrm{d}\beta_l(\tau)\right]\mathrm{d}s \\
&=\sum_{r\in\mathbb{N}^d}\frac{\sqrt{\lambda_r}\phi_r(x)}{\gamma_r}\sum_{i=0}^k\int_{t_i}^{t_{i+1}}\left(\frac{1}{\Delta t}\int_{t_i}^{t_{i+1}}[\sin(\gamma_r(t-s))-\sin(\gamma_r(t-\tau))]\mathrm{d}\tau\right)\mathrm{d}\beta_r(s).
\end{aligned}
$$

By the the orthogonal property of ϕ_r and the mutual independence of β_r, we have

$$
\begin{aligned}
&\mathbb{E}\left|\int_D B^2(t,x)\mathrm{d}x\right| \\
&=\sum_{r\in\mathbb{N}^d}\frac{\lambda_r}{\gamma_r^2}\sum_{i=0}^k\int_{t_i}^{t_{i+1}}\left|\frac{1}{\Delta t}\int_{t_i}^{t_{i+1}}[\sin(\gamma_r(t-s))-\sin(\gamma_r(t-\tau))]\mathrm{d}\tau\right|^2\mathrm{d}s \\
&\le C\sum_{r\in\mathbb{N}^d}\frac{\lambda_r}{\gamma_r^2}\sum_{i=0}^k\int_{t_i}^{t_{i+1}}\left|\frac{1}{\Delta t}\Delta t(1\wedge|r|^2|t-s|)\right|^2\mathrm{d}s \\
&\le C\sum_{r\in\mathbb{N}^d}\frac{\lambda_r}{|r|^4}(1\wedge|r|^4\Delta t^2)\le C\Delta t^{(2-\frac{d}{2}+\frac{\lambda}{2})\wedge 2}.
\end{aligned}
$$

Combining the estimates of $A^2(t,x)$ and $B^2(t,x)$ together, we obtain that

$$
\mathbb{E}\left|\int_D e^2(t,x)\mathrm{d}x\right|\le C\left(\Delta t^{(2-\frac{d}{2}+\frac{\lambda}{2})\wedge 2}+\int_0^t\mathbb{E}\left|\int_D e^2(s,x)\mathrm{d}x\right|\mathrm{d}s\right).
$$

We conclude the theorem by applying Gronwall's lemma. □

Now we approximate the integral formula (2.6) by a truncated series. Let n be a positive integer and $u_{m,n}$ be the solution of the following equation

$$
\begin{aligned}
u_{m,n}(t,x) &= \sum_{r,n}^{*}\frac{\sin(\gamma_r t)}{\gamma_r}\phi_r(x)\int_D v_0(y)\phi_r(y)\mathrm{d}y+\sum_{r,n}^{*}\cos(\gamma_r t)\phi_r(x)\int_D u_0(y)\phi_r(y)\mathrm{d}y \\
&+\sum_{r,n}^{*}\int_0^t\int_D\frac{\sin(\gamma_r(t-s))}{\gamma_r}\phi_r(x)\phi_r(y)f(s,y,u_{m,n}(s,y))\mathrm{d}s\mathrm{d}y \\
&+\sum_{r,n}^{*}\int_0^t\int_D\frac{\sin(\gamma_r(t-s))}{\gamma_r}\phi_r(x)\phi_r(y)\dot W_m(s,y)\mathrm{d}s\mathrm{d}y, \qquad (3.4)
\end{aligned}
$$

where, and throughout this section, we use the notation

$$\sum_{r,n}^{*} := \sum_{r\in\mathbb{N}^d,\max_{1\le i\le d} r_i<n}.$$

Then $u_{m,n}$ approximates u_m, the solution of (3.3), in the following way.

Theorem 3.2. *Suppose that the Lipschitz condition* (2.1) *and linear growth condition* (2.2) *are satisfied,* $v_0 \in H^{\beta,2}(D)$ *for some* $\beta > \frac{d-4}{2}$ *and* $u_0 \in H^{\alpha,2}(D)$ *for some* $\alpha > \frac{d}{2}$*, then there exists a constant* $C = C(\alpha,\beta,\lambda,T)$*, such that*

$$\sup_{t\in[0,T]} \mathbb{E}\left|\int_D |u_m(t,x)-u_{m,n}(t,x)|^2 \mathrm{d}x\right| \le C(\frac{1}{n})^{(4+2\beta-d)\wedge(2\alpha-d)\wedge(4-d)}.$$

Moreover, if $f=0$*, then*

$$\sup_{t\in[0,T]} \mathbb{E}\left|\int_D |u_m(t,x)-u_{m,n}(t,x)|^2 \mathrm{d}x\right| \le C(\frac{1}{n})^{(4+2\beta-d)\wedge(2\alpha-d)\wedge(4-d+\lambda)}.$$

Proof. Let $e(t,x) = u_m(t,x) - u_{m,n}(t,x)$, and

$$\begin{aligned}
A_1(t,x) &= \sum_{r,n}^{*} \frac{\sin(\gamma_r t)}{\gamma_r}\phi_r(x) < v_0, \phi_r >,\\
A_2(t,x) &= \sum_{r,n}^{*} \cos(\gamma_r t)\phi_r(x) < u_0, \phi_r >,\\
A_3(t,x) &= \sum_{r\in\mathbb{N}^d} \int_0^t\int_D \frac{\sin(\gamma_r(t-s))}{\gamma_r}\phi_r(x)\phi_r(y) f(s,y,u_m(s,y))\mathrm{d}s\mathrm{d}y\\
&\quad -\sum_{r,n}^{*} \int_0^t\int_D \frac{\sin(\gamma_r(t-s))}{\gamma_r}\phi_r(x)\phi_r(y) f(s,y,u_{m,n}(s,y))\mathrm{d}s\mathrm{d}y,\\
A_4(t,x) &= \sum_{r,n}^{*} \int_0^t\int_D \frac{\sin(\gamma_r(t-s))}{\gamma_r}\phi_r(x)\phi_r(y)\dot{W}_m(s,y)\mathrm{d}s\mathrm{d}y.
\end{aligned}$$

Then,

$$e(t,x) = \sum_{k=1}^{4} A_k(t,x), \quad e^2(t,x) \le 4\sum_{k=1}^{4} A_k^2(t,x).$$

By Cauchy-Schwartz inequality and the assumptions on v_0, we have

$$A_1^2(t,x) \le C\|v_0\|_{\beta,2}^2 \sum_{r,n}^{*} |r|^{-4-2\beta} \le C \sum_{r\in\mathbb{N}^d,|r|\ge n} |r|^{-4-2\beta} \le C(\frac{1}{n})^{4+2\beta-d}.$$

Similarly,

$$A_2^2(t,x) \le \|u_0\|_{\alpha,2}^2 \sum_{r,n}^{*} |r|^{-2\alpha} \le C \sum_{r\in\mathbb{N}^d,|r|\ge n} |r|^{-2\alpha} \le C(\frac{1}{n})^{2\alpha-d}.$$

Now consider the decomposition of $A_3(t,x)$,

$$\begin{aligned} A_3(t,x) =& \sum_{r,n}^{*} \int_0^t \int_D \frac{\sin(\gamma(t-s))}{\gamma_r} \phi_r(x)\phi_r(y) f(s,y,u_m(s,y)) \mathrm{d}s\mathrm{d}y \\ &+ \sum_{r,n}^{*} \int_0^t \int_D \frac{\sin(\gamma(t-s))}{\gamma_r} \phi_r(x)\phi_r(y) \\ &\times \left(f(s,y,u_m(s,y)) - f(s,y,u_{m,n}(s,y)) \right) \mathrm{d}s\mathrm{d}y \\ =:& A_{31}(t,x) + A_{32}(t,x). \end{aligned}$$

Then we have $A_3^2(t,x) \leq 2\left(A_{31}^2(t,x) + A_{32}^2(t,x)\right)$. By a similar way as in the proof of Lemma 2.2, we can derive that there exists a constant C independent of m, such that

$$\sup_{(t,x)\in[0,T]\times[0,1]} \mathbb{E}|u_m(t,x)|^2 < C. \tag{3.5}$$

By Cauchy-Schwartz inequality, the linear growth condition (2.2) and (3.5), one gets

$$\mathbb{E}\left|A_{31}(t,x)\right|^2 \leq C \sum_{r\in\mathbb{N}^d, |r|\geq n} \frac{1}{|r|^4} \leq C(\frac{1}{n})^{4-d}.$$

By the Lipschitz condition (2.1), one has

$$\mathbb{E}\left|A_{32}(t,x)\right|^2 \leq C \sum_{r\in\mathbb{N}^d, |r|\geq n} \frac{1}{|r|^4} \int_0^t \mathbb{E}\|e(s,\cdot)\|^2 \mathrm{d}s \leq C \int_0^t \mathbb{E}\|e(s,\cdot)\|^2 \mathrm{d}s.$$

For $A_4(t,x)$, without of loss of generality, we assume $t = t_{k+1}$. By the expression (3.1) of $\dot{W}_m(t,x)$ and the orthogonal property of $\phi_r(y)$, we have

$$\begin{aligned} &\mathbb{E}\left|A_4(t,x)\right|^2 \\ =& \mathbb{E}\left| \sum_{r,n}^{*} \int_0^t \int_D \frac{\sin(\gamma_r(t-s))}{\gamma_r} \phi_r(x)\phi_r(y)\dot{W}_m(s,y)\mathrm{d}s\mathrm{d}y \right|^2 \\ =& \mathbb{E}\left| \sum_{r,n}^{*} \int_0^t \frac{\sin(\gamma_r(t-s))}{\gamma_r} \phi_r(x)\sqrt{\lambda_r} \sum_{i=1}^{m} \frac{1}{\sqrt{\Delta t}} \beta_{ri}\chi_i(s)\mathrm{d}s \right|^2 \\ =& \mathbb{E}\left| \sum_{r,n}^{*} \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \frac{\sin(\gamma_r(t-s))}{\gamma_r} \phi_r(x)\sqrt{\lambda_r} \frac{1}{\sqrt{\Delta t}} \beta_{r(i+1)}\mathrm{d}s \right|^2 \\ =& \sum_{r,n}^{*} \sum_{i=0}^{k} \left| \int_{t_i}^{t_{i+1}} \frac{\sin(\gamma_r(t-s))}{\gamma_r} \phi_r(x)\sqrt{\lambda_r} \frac{1}{\sqrt{\Delta t}} \mathrm{d}s \right|^2 \\ \leq& C \sum_{r,n}^{*} \sum_{i=0}^{k} \left| |r|^{-2} \sqrt{\lambda_r} \Delta t \frac{1}{\sqrt{\Delta t}} \mathrm{d}s \right|^2 \\ \leq& C \sum_{r\in\mathbb{N}^d, |r|\geq n} |r|^{-4-\lambda} \leq C(\frac{1}{n})^{4-d+\lambda}. \end{aligned}$$

Combining the estimation of $A_i^2(t,x),(i=1,2,3,4)$ together, we obtain

$$\mathbb{E}\left|\int_D e^2(t,x)\mathrm{d}x\right| \le C\left((\frac{1}{n})^{(4+2\beta-d)\wedge(2\alpha-d)\wedge(4-d)} + \int_0^t \mathbb{E}\left|\int_D e^2(s,x)\mathrm{d}x\right|\mathrm{d}s\right).$$

Moreover, if $f=0$, then $A_3=0$, thus in this case

$$\mathbb{E}\left|\int_D e^2(t,x)\mathrm{d}x\right| \le C\left((\frac{1}{n})^{(4+2\beta-d)\wedge(2\alpha-d)\wedge(4-d+\lambda)} + \int_0^t \mathbb{E}\left|\int_D e^2(s,x)\mathrm{d}x\right|\mathrm{d}s\right).$$

We conclude the theorem by applying Gronwall's lemma. □

4. Difference scheme and error analysis

In this section, an explicit difference scheme is used to approximate $u_{m,n}(t,x)$, and the error between the numerical solution and the solution $u(t,x)$ of (1.1) is analyzed.

Let n be the positive integer that has been used in Section 3 to truncate integral formula (2.6). Define a partition of $D=[0,1]^d$ by

$$x_k = (k_1\Delta x, k_2\Delta x, \dots, k_d\Delta x),$$

where $\Delta x = \frac{1}{n}$ and

$$k \in K = \{r=(r_1,\dots,r_d)\in\mathbb{N}^d : 1\le r_i\le n-1, 1\le i\le d\}.$$

Define

$$\bar{K} = \big\{r=(r_1,\dots,r_d)\in\mathbb{N}^d : 0\le r_i\le n-1, 1\le i\le d\big\}.$$

For $l\in\bar{K}$, let

$$[x_l, x_{l+1}] = \big\{x=(x_1,\dots,x_d)\in\mathbb{R}^d : l_i\Delta x\le x_i\le (l_i+1)\Delta x, 1\le i\le d\big\}.$$

We consider the difference solution $u_{m,n}^n$ defined on the lattice $\{(t_i,x_k); i\in\{0,1,\dots,m\}, k\in K\}$, as

$$\begin{aligned}
u_{m,n}^n(t_{i+1},x_k) &= \frac{1}{n^d}\sum_{l\in K}\sum_{r\in K}\frac{\sin(\gamma_r t_{i+1})}{\gamma_r}\phi_r(x_k)\phi_r(x_l)v_0(x_l)\\
&+\frac{1}{n^d}\sum_{l\in K}\sum_{r\in K}\cos(\gamma_r t_{i+1})\phi_r(x_k)\phi_r(x_l)u_0(x_l)\\
&+\sum_{q=0}^{i}\sum_{l\in\bar{K}}\int_{t_q}^{t_{q+1}}\int_{[x_l,x_{l+1}]}\sum_{r\in K}\frac{\sin(\gamma_r(t_{i+1}-s))}{\gamma_r}\phi_r(x_k)\phi_r(y)f(t_q,x_l,u_{m,n}^n(t_q,x_l))\mathrm{d}s\mathrm{d}y\\
&+\sum_{q=1}^{i+1}\int_{t_{q-1}}^{t_q}\sum_{r\in K}\frac{\sin(\gamma_r(t_{i+1}-s))}{\gamma_r}\phi_r(x_k)\sqrt{\lambda_r}\frac{\beta_{rq}}{\sqrt{\Delta t}}\mathrm{d}s. \qquad (4.1)
\end{aligned}$$

The following theorem gives the error in the difference method given by (4.1).

Theorem 4.1. *Assume the conditions in theorem* 3.1 *and* 3.2 *are all satisfied,* $v_0 \in H^{\beta,2}(D)$ *with* $\beta > \frac{d}{2}$ *and* $u_0 \in H^{\alpha,2}(D)$ *with* $\alpha > d$*, then there exists a constant* $C = C(\alpha,\beta,\lambda,T)$*, such that*

$$\begin{aligned}&\sup_{0\le i\le m} \mathbb{E}\left|\frac{1}{n^d}\sum_{k\in K}|u_{m,n}(t_i,x_k) - u^n_{m,n}(t_i,x_k)|^2\right| \\ &\le C\left(\Delta t^{(2+\beta-\frac{d}{2})\wedge(\alpha-\frac{d}{2})\wedge(2-\frac{d}{2})} + \Delta x^{(2\beta-d)\wedge(2\alpha-2d)\wedge(4-d)\wedge 2}\right).\end{aligned}$$

Moreover, if $f = 0$*, then*

$$\sup_{0\le i\le m} \mathbb{E}\left|\frac{1}{n^d}\sum_{k\in K}|u_{m,n}(t_i,x_k) - u^n_{m,n}(t_i,x_k)|^2\right| \le C\Delta x^{(2\beta-d)\wedge(2\alpha-2d)}.$$

Proof. It is clear that $u_{m,n}(t_{i+1},x_k)$ satisfies

$$\begin{aligned}u_{m,n}(t_{i+1},x_k) &= \sum_{r\in K}\frac{\sin(\gamma_r t_{i+1})}{\gamma_r}\phi_r(x_k)\int_D v_0(y)\phi_r(y)\mathrm{d}y \\ &+ \sum_{r\in K}\cos(\gamma_r t_{i+1})\phi_r(x_k)\int_D u_0(y)\phi_r(y)\mathrm{d}y \\ &+ \sum_{r\in K}\int_0^{t_{i+1}}\int_D \frac{\sin(\gamma_r(t_{i+1}-s))}{\gamma_r}\phi_r(x_k)\phi_r(y)f(s,y,u_{m,n}(s,y))\mathrm{d}s\mathrm{d}y \\ &+ \sum_{r\in K}\int_0^{t}\int_D \frac{\sin(\gamma_r(t_{i+1}-s))}{\gamma_r}\phi_r(x_k)\phi_r(y)\dot{W}_m(s,y)\mathrm{d}s\mathrm{d}y \\ =:& A_1(t_{i+1},x_k) + A_2(t_{i+1},x_k) + A_3(t_{i+1},x_k) + A_4(t_{i+1},x_k).\end{aligned}$$

For $y = (y_1,\dots,y_d) \in D, t \in [0,T]$, set $k_n(y) = (\frac{[ny_1]}{n},\frac{[ny_2]}{n},\dots,\frac{[ny_d]}{n})$ and $k_{mT}(t) = \frac{T}{m}[\frac{mt}{T}]$, then the difference scheme (4.1) can be written as

$$\begin{aligned}u^n_{m,n}(t_{i+1},x_k) &= \sum_{r\in K}\frac{\sin(\gamma_r t_{i+1})}{\gamma_r}\phi_r(x_k)\int_D v_0(k_n(y))\phi_r(k_n(y))\mathrm{d}y \\ &+ \sum_{r\in K}\cos(\gamma_r t_{i+1})\phi_r(x_k)\int_D u_0(k_n(y))\phi_r(k_n(y))\mathrm{d}y \\ &+ \sum_{r\in K}\int_0^{t_{i+1}}\int_D \frac{\sin(\gamma_r(t_{i+1}-s))}{\gamma_r}\phi_r(x_k)\phi_r(y) \\ &\times f(k_{mT}(s),k_n(y),u^n_{m,n}(k_{mT}(s),k_n(y)))\mathrm{d}s\mathrm{d}y \\ &+ \sum_{q=1}^{i+1}\int_{t_{q-1}}^{t_q}\sum_{r\in K}\frac{\sin(\gamma_r(t_{i+1}-s))}{\gamma_r}\phi_r(x_k)\sqrt{\lambda_r}\frac{\beta_{rq}}{\sqrt{\Delta t}}\mathrm{d}s \\ =:& B_1(t_{i+1},x_k) + B_2(t_{i+1},x_k) + B_3(t_{i+1},x_k) + B_4(t_{i+1},x_k).\end{aligned}$$

Let $e_{i,k}=u_{m,n}(t_i,x_k)-u^n_{m,n}(t_i,x_k)$. It is clear that

$$e_{i+1,k}=\sum_{j=1}^{4}\Big(A_j(t_{i+1},x_k)-B_j(t_{i+1},x_k)\Big).$$

Thus

$$\mathbb{E}|e_{i+1,k}|^2\le 4\sum_{j=1}^{4}\mathbb{E}\Big|A_j(t_{i+1},x_k)-B_j(t_{i+1},x_k)\Big|^2.$$

As mentioned in [14, 27], the $(n-1)^d$-dimensional vectors

$$\nu_r=\left(\sqrt{\frac{1}{n^d}}\phi_r(x_k),k\in K\right),\quad r\in K, \tag{4.2}$$

are an orthonormal basis of $\mathbb{R}^{(n-1)d}$. That is to say,

$$\int_D\phi_r(k_n(y))\phi_l(k_n(y))\mathrm{d}y=\delta_{rl},r,l\in K, \tag{4.3}$$

where δ_{rl} is the Kronecher symbol. Then, it holds

$$\begin{aligned}
&|A_1(t_{i+1},x_k)-B_1(t_{i+1},x_k)|^2\\
=&\left|\sum_{r\in K}\frac{\sin(\gamma_r t_{i+1})}{\gamma_r}\phi_r(x_k)\int_D v_0(y)\phi_r(y)-v_0(k_n(y))\phi_r(k_n(y))\mathrm{d}y\right|^2\\
=&\left|\sum_{r\in K}\frac{\sin(\gamma_r t_{i+1})}{\gamma_r}\phi_r(x_k)\left(\int_D v_0(y)\phi_r(y)\mathrm{d}y\right.\right.\\
&\qquad\left.\left.-\sum_{l\in\mathbb{N}^d}<v_0,\phi_l>\int_D\phi_l(k_n(y))\phi_r(k_n(y))\mathrm{d}y\right)\right|^2\\
=&\left|\sum_{r\in K}\frac{\sin(\gamma_r t_{i+1})}{\gamma_r}\phi_r(x_k)\sum_{l\in\mathbb{N}^d-K}<v_0,\phi_l>\int_D\phi_l(k_n(y))\phi_r(k_n(y))\mathrm{d}y\right|^2\\
\le& C\sum_{l\in\mathbb{N}^d-K}|<v_0,\phi_l>|^2|l|^{2\beta}\sum_{l\in\mathbb{N}^d-K}|l|^{-2\beta}\left|\sum_{r\in K}\frac{\sin(\gamma_r t_{i+1})}{\gamma_r}\phi_r(x_k)\right|^2\\
\le& C\|v_0\|^2_{\beta,2}\sum_{|l|\ge n}|l|^{-2\beta}\le C(\frac{1}{n})^{2\beta-d}.
\end{aligned}$$

Using similar arguments, one can obtain that

$$|A_2(t_{i+1},x_k)-B_2(t_{i+1},x_k)|^2\le C(\frac{1}{n})^{2(\alpha-d)}.$$

By Cauchy-Schwartz inequality, one has

$$
\begin{aligned}
&\mathbb{E}\left|A_3(t_{i+1},x_k)-B_3(t_{i+1},x_k)\right|^2\\
=&\Big|\sum_{r\in K}\int_0^{t_{i+1}}\int_D \frac{\sin(\gamma_r(t_{i+1}-s))}{\gamma_r}\phi_r(x_k)\phi_r(y)\\
&\quad\times(f(s,y,u_{m,n}(s,y))-f(k_{mT}(s),k_n(y),u^n_{m,n}(k_{mT}(s),k_n(y))))\mathrm{d}s\mathrm{d}y\Big|^2\\
=&\Big|\sum_{r\in K}\int_0^{t_{i+1}}\int_D \frac{\sin(\gamma_r(t_{i+1}-s))}{\gamma_r}\phi_r(x_k)\phi_r(y)\\
&\quad\times(f(s,y,u_{m,n}(s,y))-f(k_{mT}(s),k_n(y),u_{m,n}(k_{mT}(s),k_n(y))))\mathrm{d}s\mathrm{d}y\Big|^2\\
&\quad+\Big|\sum_{r\in K}\int_0^{t_{i+1}}\int_D \frac{\sin(\gamma_r(t_{i+1}-s))}{\gamma_r}\phi_r(x_k)\phi_r(y)\\
&\quad\times\big(f(k_{mT}(s),k_n(y),u_{m,n}(k_{mT}(s),k_n(y)))\\
&\quad-f(k_{mT}(s),k_n(y),u^n_{m,n}(k_{mT}(s),k_n(y)))\big)\,\mathrm{d}s\mathrm{d}y\Big|^2\\
=:&\,A_{31}+A_{32}.
\end{aligned}
$$

Using the same method as in Lemma 2.3, we can obtain

$$
\begin{aligned}
&\mathbb{E}|u_{m,n}(t,x)-u_{m,n}(s,y)|^2\\
\le&\, C\left(|t-s|^{(\beta-\frac{d-4}{2})\wedge(\alpha-\frac{d}{2})\wedge(2-\frac{d}{2})}+|x-y|^{(2\beta-(d-4))\wedge(2\alpha-d)\wedge(4-d)\wedge 2}\right),
\end{aligned}
\tag{4.4}
$$

for every $s,t\in[0,T]$ and $x,y\in D$. By (4.4) and the Lipschitz condition (2.1), one has

$$
\begin{aligned}
A_{31}\le&\, C\sum_{r\in K}\frac{1}{|r|^4}\Big(\Delta t^2+\Delta x^2+\mathbb{E}|u_{m,n}(s,y)-u_{m,n}(k_{mT}(s),k_n(y))|^2\Big)\\
\le&\, C\Big(\Delta t^{(\beta-\frac{d-4}{2})\wedge(\alpha-\frac{d}{2})\wedge(2-\frac{d}{2})}+\Delta x^{(2\beta-(d-4))\wedge(2\alpha-d)\wedge(4-d)\wedge 2}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
A_{32}\le&\, C\mathbb{E}\left|\int_0^{t_{i+1}}\int_D\left|u_{m,n}(k_{mT}(s),k_n(y))-u^n_{m,n}(k_{mT}(s),k_n(y))\right|^2\mathrm{d}s\mathrm{d}y\right|\\
=&\, C\Delta t\sum_{q=0}^{i}\mathbb{E}\left|\frac{1}{n^d}\sum_{\in K}|e_{q,k}|^2\right|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\mathbb{E}\left|A_3(t_{i+1},x_k)-B_3(t_{i+1},x_k)\right|^2\\
\le&\, C\Bigg(\Delta t^{(\beta-\frac{d-4}{2})\wedge(\alpha-\frac{d}{2})\wedge(2-\frac{d}{2})}+\Delta x^{(2\beta-(d-4))\wedge(2\alpha-d)\wedge(4-d)\wedge 2}\\
&\quad+\Delta t\sum_{q=0}^{i}\mathbb{E}\left|\frac{1}{n^d}\sum_{\in K}|e_{q,k}|^2\right|\Bigg).
\end{aligned}
$$

By the orthogonal property ϕ_r, there holds

$$\begin{aligned}
A_4(t_{i+1},x_k) &= \sum_{r\in K}\int_0^t\int_D \frac{\sin(\gamma_r(t_{i+1}-s))}{\gamma_r}\phi_r(x_k)\phi_r(y)\dot{W}_m(s,y)\mathrm{d}s\mathrm{d}y \\
&= \sum_{q=1}^{i+1}\int_{t_{q-1}}^{t_q}\sum_{r\in K}\frac{\sin(\gamma_r(t_{i+1}-s))}{\gamma_r}\phi_r(x_k) \\
&\qquad \times \sum_{l\in K}\sqrt{\lambda_l}\frac{\beta_{lq}}{\sqrt{\Delta t}}\int_D \phi_r(y)\phi_l(y)\mathrm{d}y\mathrm{d}s \\
&= \sum_{q=1}^{i+1}\int_{t_{q-1}}^{t_q}\sum_{r\in K}\frac{\sin(\gamma_r(t_{i+1}-s))}{\gamma_r}\phi_r(x_k)\sqrt{\lambda_r}\frac{\beta_{rq}}{\sqrt{\Delta t}}\mathrm{d}s \\
&= B_4(t_{i+1},x_k).
\end{aligned}$$

Thus we have

$$\mathbb{E}|A_4(t_{i+1},x_k)-B_4(t_{i+1},x_k)|^2 = 0.$$

Combining together the above arguments, we can obtain that

$$\begin{aligned}
\mathbb{E}\left|\frac{1}{n^d}\sum_{k\in K}|e_{i+1,k}|^2\right| \le C\bigg(&\Delta t^{(\beta-\frac{d-4}{2})\wedge(\alpha-\frac{d}{2})\wedge(2-\frac{d}{2})} \\
&+\Delta x^{(2\beta-d)\wedge(2\alpha-2d)\wedge(4-d)\wedge 2} + \Delta t\sum_{q=0}^{i}\mathbb{E}\left|\frac{1}{n^d}\sum_{k\in K}|e_{q,k}|^2\right|\bigg).
\end{aligned}$$

Moreover, if $f=0$, then $A_3=B_3=0$, thus in this case

$$\mathbb{E}\left|\frac{1}{n^d}\sum_{k\in K}|e_{i+1,k}|^2\right| \le C\Delta x^{(2\beta-d)\wedge(2\alpha-2d)}.$$

We conclude the theorem by using the Gronwall's inequality of discrete form. □

By Theorems 3.1, 3.2, 4.1 and Lemma 2.3, we have the following theorem.

Theorem 4.2. *Assume the conditions in Theorem* 4.1 *are all satisfied. Then there exists a constant* $C=C(\alpha,\beta,\lambda,T,K,L)$, *such that*

$$\begin{aligned}
&\sup_{0\le i\le m}\mathbb{E}\left|\frac{1}{n^d}\sum_{k\in K}|u(t_i,x_k)-u^n_{m,n}(t_i,x_k)|^2\right| \\
&\le C\Big(\Delta t^{(2+\beta-\frac{d}{2})\wedge(\alpha-\frac{d}{2})\wedge(2-\frac{d}{2})} + \Delta x^{(2\beta-d)\wedge(2\alpha-2d)\wedge(4-d)\wedge 2}\Big).
\end{aligned}$$

Moreover, if the nonlinear term $f=0$, then

$$\begin{aligned}&\sup_{0\le i\le m}\mathbb{E}\left|\frac{1}{n^d}\sum_{k\in K}|u(t_i,x_k)-u^n_{m,n}(t_i,x_k)|^2\right|\\ \le\, &C\Big(\Delta t^{(2+\beta-\frac{d}{2})\wedge(\alpha-\frac{d}{2})\wedge(2-\frac{d}{2}+\frac{\lambda}{2})\wedge 2}+\Delta x^{(2\beta-d)\wedge(2\alpha-2d)\wedge(4-d+\lambda)\wedge 2}\Big).\end{aligned}$$

5. Numerical results

The numerical methods described in the above sections are computationally tested in this section. Notice that by Theorem 4.2, for sufficiently smooth initial conditions the rate of convergence in l^2 of the difference scheme with respect to x is of order $1\wedge(2-\frac{d}{2})$ for fixed temporal step length, and with respect to t is of order $1-\frac{d}{4}$ for fixed spacial step length. We now test this statement by the following examples.

Example 5.1. Consider Eq. (1.1) with $d=1$, $T=1$, $u_0(x)=\sin(\pi x), v_0(x)=\sin(2\pi x)$, $f(t,x,u)=1+\cos(t)+\sin(x)+u+\arctan(u)$, and $\lambda_k=1,(k=1,2,\ldots)$.

Here, the exact solution u is approximated by the explicit difference method (4.1) with a very small step size, $\Delta x=\frac{1}{n_{exact}}=2^{-8}$ and $\Delta t=\frac{1}{m_{exact}}=2^{-9}$. The expected values are approximated by computing averages over $M=100$ times.

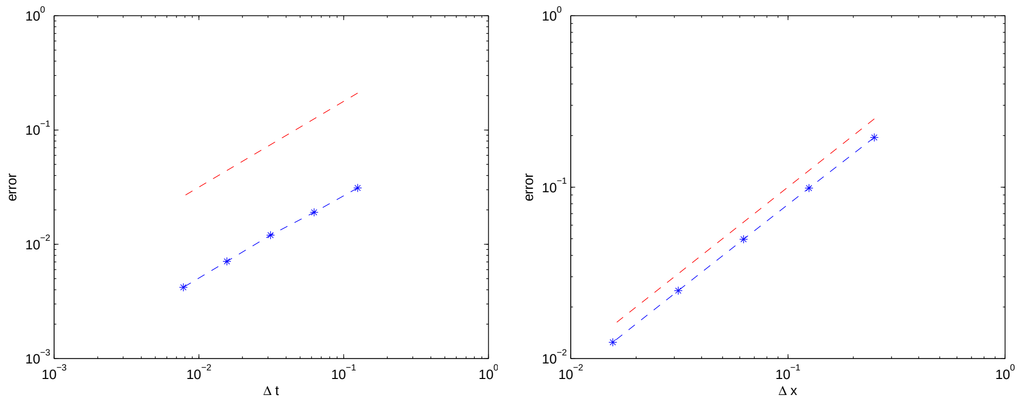

Figure 1: l^2-convergence for Example 5.1 with respect to t and x.

First, we fix $n=2^8$ and let m changes from 2^3 to 2^7. The MATLAB command loglog plots our approximation to $(\mathbb{E}|\frac{1}{n}\sum_{k=1}^{n-1}|u(1,x_k)-u^n_{m,n}(1,x_k)|^2|)^{\frac{1}{2}}$ against Δt on a log-log scale. This produces the asterisks connected with solid lines in the left-hand plot of Fig. 1. A dashed line of slope 0.75 is added. We see that the slopes of the two curves appear to match well. We test this further by assuming that

$$\left(\mathbb{E}\left|\frac{1}{n}\sum_{k=1}^{n-1}|u(1,x_k)-u^n_{m,n}(1,x_k)|^2\right|\right)^{\frac{1}{2}}=C(\Delta t)^q$$

for fixed sufficiently small Δx, so that

$$\log\left(\mathbb{E}\Big|\frac{1}{n}\sum_{k=1}^{n-1}|u(1,x_k)-u^n_{m,n}(1,x_k)|^2\Big|\right)^{\frac{1}{2}}=\log C+q\log(\Delta t).$$

A least squares fit for $\log C$ and q producing the value $q=0.7465$ with a least squares residual of 0.1214. Hence, the computational results are consistent with the order of convergence with respect to t equal to 0.75.

Second, we fix $m=2^9$ and let n changes from 2^2 to 2^6. The MATLAB command loglog plots our approximation to $(\mathbb{E}|\frac{1}{n}\sum_{k=1}^{n-1}|u(1,x_k)-u^n_{m,n}(1,x_k)|^2|)^{\frac{1}{2}}$ against Δx on a log-log scale. This produces the asterisks connected with solid lines in the right-hand plot of Fig. 1. For a reference, a dashed line of slope 1 is added. We see that the slopes of the two curves appear to match well. As the case for Δt, a least squares fit for $\log C$ and q producing the value $q=0.9905$ with a least squares residual of 0.01011. Hence, the computational results are consistent with the order of convergence with respect to x equal to 1.0.

Example 5.2. Consider Eq. (1.1) with $d=1$, $T=1$, $u_0(x)=\sin(\pi x), v_0(x)=\sin(2\pi x), f=0$, and $\lambda_k=\frac{1}{k},(k=1,2,\dots)$.

Since $f=0$ and the initial conditions are smooth, by Theorem 4.2, the rate of convergence in l^2 of the difference scheme with respect to x is of order $1\wedge(2-\frac{d}{2}+\frac{\lambda}{2})=1$ for fixed temporal step length, and with respect to t is of order $1\wedge(1-\frac{d}{4}+\frac{\lambda}{4})=1$ for fixed spacial step length.

Here, the exact solution u is approximated by the explicit difference method (4.1) with a very small step size, $\Delta x=\frac{1}{n_{exact}}=2^{-9}$ and $\Delta t=\frac{1}{m_{exact}}=2^{-10}$. The expected values are approximated by computing averages over $M=100$ times.

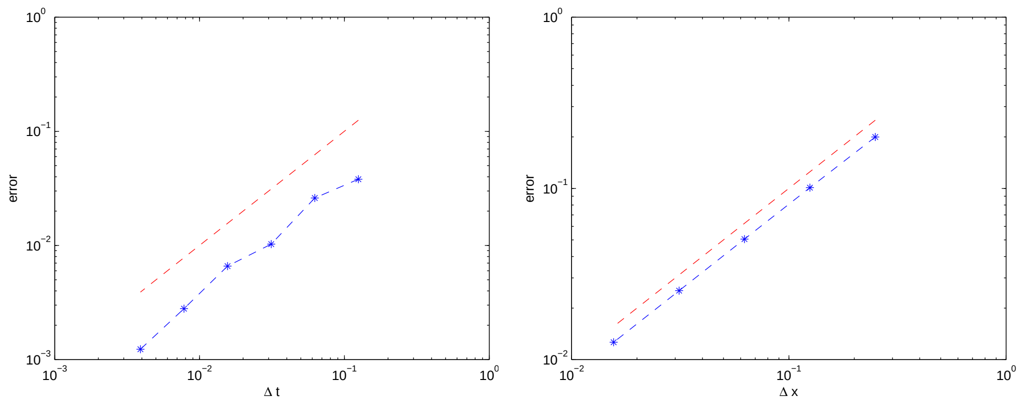

Figure 2: l^2-convergence for Example 5.2 with respect to t and x.

First, we fix $n=2^9$ and let m changes from 2^3 to 2^8. The MATLAB command loglog plots our approximation to $(\mathbb{E}|\frac{1}{n}\sum_{k=1}^{n-1}|u(1,x_k)-u^n_{m,n}(1,x_k)|^2|)^{\frac{1}{2}}$ against Δt on a log-log scale. This produces the asterisks connected with solid lines in the left-hand plot of Fig. 2. A dashed line of slope 1 is added. We see that the slopes of the two curves appear to match well. We test this further by assuming that

$$\left(\mathbb{E}\Big|\frac{1}{n}\sum_{k=1}^{n-1}|u(1,x_k)-u^n_{m,n}(1,x_k)|^2\Big|\right)^{\frac{1}{2}}=C(\Delta t)^q$$

for fixed sufficiently small Δx, so that

$$\log\left(\mathbb{E}\Big|\frac{1}{n}\sum_{k=1}^{n-1}|u(1,x_k)-u^n_{m,n}(1,x_k)|^2\Big|\right)^{\frac{1}{2}}=\log C+q\log(\Delta t).$$

A least squares fit for $\log C$ and q producing the value $q=1.0009$ with a least squares residual of 0.3141. Hence, the computational results are consistent with the order of convergence with respect to t equal to 1.

Second, we fix $m=2^{10}$ and let n changes from 2^2 to 2^6. The MATLAB command loglog plots our approximation to $(\mathbb{E}|\frac{1}{n}\sum_{k=1}^{n-1}|u(1,x_k)-u^n_{m,n}(1,x_k)|^2|)^{\frac{1}{2}}$ against Δx on a log-log scale. This produces the asterisks connected with solid lines in the right-hand plot of Fig. 2. For a reference, a dashed line of slope 1 is added. We see that the slopes of the two curves appear to match well. As the case for Δt, a least squares fit for $\log C$ and q producing the value $q=0.9960$ with a least squares residual of 0.0067. Hence, the computational results are consistent with the order of convergence with respect to x equal to 1.0.

6. Conclusions

An explicit difference method for stochastic elastic equations driven by infinite dimensional noise are investigated. Our main results Theorem 4.2 showed that for sufficiently smooth initial conditions, the rate of convergence of the difference scheme with respect to x is of order $1\wedge(2-\frac{d}{2})$, while with respect to t is of order $1-\frac{d}{4}$. Numerical experiments showed that the theoretical analyses for the order of convergence were correct and the methods were computationally feasible. In this paper we only studied the case of additive noise. We will investigate the case of multiplicative noise in our future works.

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References

[1] E. J. Allen, S. J. Novosel, and Z. Zhang, *Finite element and difference approximation of some linear stochastic partical differential equations*, Stoch. Stoch. Rep., vol. 64 (1998), pp. 117–142.

[2] V. Bally and D. Talay, *The law of the Euler scheme for stochastic differential equations. I. Convergence rate of the distribution function*, Probab. Theory Relat. Field., vol. 104 (1996), pp. 43–60.
[3] Z. Brzeźniak, B. Maslowski and J. Seidler, *Stochastic nonlinear beam equations*, Probab. Theory Relat. Field., vol. 132 (2005), pp. 119–149.
[4] Y. Z. Cao, H. T. Yang and L. Yin, *Finite element methods for semilinear elliptic stochastic partial differential equations*, Numer. Math., vol. 106 (2007), pp. 181–198.
[5] Y. Z. Cao, R. Zhang and K. Zhang, *Finite element and discontinuous Galerkin method for stochastic Helmholtz equation in two and three dimensions*, J. Comput. Math., vol. 5 (2008), pp. 702–715.
[6] P. L. Chow and J. L. Menaldi, *Stochastic PDE for nonlinear Vibration of elastic panels*, Diff. Int. Eqns., vol. 12 (1999), pp. 419–434.
[7] A. Debussche and J. Printems, *Weak order for the discretization of the stochastic heat equation*, Math. Comput., vol. 78 (2009), pp. 845–863.
[8] Q. Du and T. Y. Zhang, *Numerical approximation of some linear stochastic partial differential equations driven by special additive noise*, SIAM J. Numer. Anal., vol. 4 (2002), pp. 1421–1445.
[9] A. Etheridge, *Stochastic Partial Differential Equations*, Cambridge University Press, Cambridge, 1995.
[10] J. F. Feng, G. Y. Lei and M. P. Qian, *Second-order methods for solving stochastic differential equations*, J. Comput. Math., vol. 10 (1992), pp. 376–387.
[11] O. H. Galal, M. A. El-Tawil and A. A. Mahmoud, *Stochastic beam equations under random dynamic loads*, Inter. J. Solid. Struct., vol. 39 (2002), pp. 1031–1040.
[12] T. C. Gard, *Intruduction to Stochastic Differential Equations*, Marcel Decker, New York, 1988.
[13] I. Gyöngy and D. Nualart, *Implicit scheme for stochastic parabolic partial differential equations driven by space-time white noise*, Potential Anal., vol. 7 (1997), pp. 725–757.
[14] I. Gyöngy, *Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise I*, Potential Anal., vol. 9 (1998), pp. 1–25.
[15] I. Gyöngy and T. Mart ´ınez, *On numerical solution of stochastic partial differential equations of elliptic type*, Stochatics, vol. 78 (2006), pp. 213–231.
[16] A. Jentzen and P. E. Kloeden, *The numerical approximation of stochastic partical differential equations*, Milan J. Math., vol. 77 (2009), pp. 205–244.
[17] J. U. Kim, *On a stochatic plate equation*, Appl. Math. Optim., vol. 44 (2001), pp. 33–48.
[18] P. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, Berlin, 1992.
[19] P. Kloeden, E. Platen, and N. Hoffmann, *Extrapolation methods for the weak approximation to Itô diffusions*, SIAM J. Numer. Anal., vol. 32 (1995), pp. 1519–1534.
[20] G. T. Kossioris and G. E. Zouraris, *Fully-discrete finite element approximations for a fourth-order linear stochastic parabolic equation with additive space-time white noise: II. 2D and 3D Case*, arXiv:0906.1828v1 [math.NA]
[21] M. Kovács, S. Larsson and F. Lindgren, *Strong convergence of the finite element method with truncated noise for semilinear parabolic stochastic equations*, Numer. Algor., vol. 53 (2010), pp. 309–320.
[22] A. Millet and Pierre-Luc Morine, *On implicit and explicit discretization schemes for parabolic SPDEs in any dimension*, Stoch. Proc. Appl., vol. 115 (2005), pp. 1073–1106.
[23] A. Martin, S. M. Prigarin and G. Winkler, *Exact and fast numerical algorithms for the stochastic wave equation*, Int. J. Comput. Math., vol. 80 (2003), pp. 1535–1541.

[24] G. D. PRATO AND J. ZABCZYK, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, UK, 1992.
[25] C. PRéVôT AND M. RöCKNER, *A Concise Course on Stochastic Differential Equations*, Lecture Notes in Mathmatics, Springer, Berlin, 2007.
[26] J. PRINTEMS, *On the discretization in time of parabolic stochastic partial differential equtions*, Math. Model. Numer. Anal., vol. 35 (2001), pp. 1055-1078.
[27] L. QUER-SARDANYONS AND M. SANZ-SOLÉ, *Space semi-discretizations for a stochastic wave equation*, Potential Anal., vol. 24 (2006), pp. 303–332.
[28] B. L. ROZOVSKII, *Stochastic evolution systems. Linear Theory and Applications to Nonlinear Filtering*, Kluwer Academic Publishers Group, Dordrecht, 1990.
[29] D. TALAY AND L. TUBARO, *Extension of the global error for numerical schemes solving stochastic differential equations*, Stoch. Anal. Appl., vol. 8 (1990), pp. 483–509.
[30] J. B. WALSH, *An Introduction to Stochastic Partial Differential Equations*, Lecture Notes in Mathematics, 1180, Springer-verlag, Berlin, (1986), pp. 265–439.
[31] Y. YAN, *Galerkin finite element methods for stochastic parabolic partial differential equations*, SIAM J. Numer. Anal., vol. 43 (2005), pp. 1363–1384.
[32] H. YOO, *Semi-discretization of stochastic partial differential equations on* $\mathbb{R}^1$ *by a finite-difference method*, Math. Comput., vol. 69 (2000), pp. 653–666.
[33] T. ZHANG, *Large deviations for stochastic nonlinear beam equations*, J. Funct. Anal., vol. 248 (2007), pp. 175–201.